N ANALOGUE OF RAMANUJAN'S REMARKABLE PRODUCT OF THETA-FUNCTIONS AND ITS APPLICATIONS

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Abstract: In this paper, we define an analogue of Ramanujan's remarkable product $l_{p,n}$ of theta functions ϕ (q) and ψ (q). We establish some general theorems for the explicit evaluations of the product same product form. We also associate the product with Ramanujan's cubic continued fraction G (q) with some illustrations.

Keywords: Theta Functions, Class Invariant, Continued Fraction.

Introduction: Ramanujan in his notebooks [2], [5] and [9] has developed the theory of theta functions in his own notation. They are defined as follows:

For a and q complex number with |q|<1

$(a)_\infty := (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$

And $(a)_n := (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) = \frac{(a)_\infty}{(aq^n)_\infty}$,

n : any integer,

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad (1.1)$$

$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty,\ |ab|<1. \quad (1.2)$$

Identity (1.2) is the Jacobi's triple product identity in Ramanujan's notation [5, Ch.16, Entry 19]. It follows from (1.1) and (1.2) that

[5, Ch.16, Entry 22],

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}. \quad (1.3)$$

$$\psi(q) := f(q,q^2) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2,q^2)_\infty}{(q;q^2)_\infty}. \quad (1.4)$$

$$\text{And}\quad \chi(q) = (-q;q^2)_\infty. \quad (1.5)$$

On page 338 of his notebook [9], Ramanujan has defined the remarkable product of theta function

$$a_{m,n} = e^{-\left(\frac{\pi}{4}\right)(n-1)\sqrt{\frac{m}{n}}}\frac{\psi^2\left(e^{-\pi\sqrt{mn}}\right)\varphi^2\left(-e^{-2\pi\sqrt{mn}}\right)}{\psi^2\left(e^{-\pi\sqrt{\frac{m}{n}}}\right)\varphi^2\left(-e^{-2\pi\sqrt{mn}}\right)}. \quad (1.6)$$

where m and n are positive real numbers. He also listed eighteen values of $a_{m,n}$ which was proved by Berndt.et.al. [1]. Again, Mahadeva Naika et. al. in [7] has defined

$$b_{m,n} = ne^{-(\pi/4)(n-1)\sqrt{m/n}} \times \frac{\psi^2(-e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(-e^{-\pi\sqrt{m/n}})\varphi^2(-e^{-2\pi\sqrt{mn}})}. \quad (1.7)$$

Mahadeva Naika et. al. [8] has established many general formulas for explicit evaluations of Ramanujan's cubic continued fraction G (q) in terms of $a_{m,n}$ and $b_{m,n}$, where

$$G(q) := \frac{q^{1/3}}{1\ +}\ \frac{q+q^2}{1+}\ \frac{q^2+q^4}{1\ +}\ \frac{q^3+q^6}{1\ +\cdots},\quad |q|<1, \quad (1.8)$$

Motivated by these works, in this paper, we define the product function $l_{p,n}$ as

$$l_{p,n} = \frac{\psi(-q^p)\psi(q)}{q^{(p-1)/8}\psi(-q)\varphi(q^p)},\ q = e^{-\pi\sqrt{n/p}}. \quad (1.9)$$

where n and p are positive real numbers.

In Section 2 we discuss few properties of $l_{p,n}$ .Section 3 contains explicit evaluations of $l_{p,n}$ in terms of Ramanujan's class invariants. In Section 4, we give some particular numerical values of $l_{p,n}$ and in Section 5 we establish some formulas for explicit evaluations of G (q) in terms of $l_{p,n}$.

To end this introduction, we present some preliminary results.

Let K, K', L and L' denote the complete integrals of the first kind associated with the moduli k, k', l and l' respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L}. \quad (1.10)$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l induced by (1.10). Ramanujan recorded his modular equations in terms of α and β where $\alpha=k^2$ and $\beta= l^2$.The multiplier m connecting α and β is defined by m=z_1/z_n where $z_r=\varphi^2(q^r)$ for q=exp (-$\pi K'/K$), |q|<1.

Lemma 1 [5, Entry 27(i), p.43] If $\alpha\beta$=π, then

$$\sqrt{\pi}\phi(e^{-\alpha^2}) = \sqrt{\beta}\phi(e^{-\beta^2}). \quad (1.11)$$

Lemma 2 [3, (1.13), p.1049] If αβ=π^2, then $e^{-\alpha/8}\sqrt[4]{\alpha}$ ψ (-$-e^{-\alpha}$) =$e^{-\beta/8}\sqrt[4]{\beta}$ ψ (-$e^{-\beta}$). (1.12)

Lemma 3 [5, Entry 10 (i), p.122] One has ϕ (q)=$\sqrt{z_1}$. (1.13)

Lemma 4 [5, Entry 11 (ii), p.123] One has

$$\psi(-q) = \sqrt{\frac{z_1}{2}}\{\alpha(1-\alpha)\}^{1/8}q^{-1/8}. \quad (1.14)$$

Lemma 5 [5, Entry 12 (v), p.124] One has

$$\chi(q) = 2^{1/4}\left\{\frac{\alpha(1-\alpha)}{q}\right\}^{-1/24},$$

$$\chi(-q) = 2^{1/6}(1-\alpha)^{1/12}\left(\frac{\alpha}{q}\right)^{-1/24}. \quad (1.15)$$

Note: Replacing q by q^n in Lemma 3 and 4, then z_1 and α will be replaced by z_n and β where β has degree n over α.

Lemma 6 [5, Entry 1 (i), (ii), p.345]

$$1+\frac{1}{G(q)} = \frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)}. \quad (1.16)$$

$$1+\frac{1}{G^3(q)} = \frac{\psi^4(q)}{q\,\psi^4(q^3)}. \quad (1.17)$$

$$2G(q) = 1 - \frac{\varphi(-q^{1/3})}{\varphi(-q^3)}. \quad (1.18)$$

Lemma 7 [5, p.347] If G (q) is defined by (1.8), then

$$8G^3(q) = \frac{\varphi^4(-q)}{\varphi^4(-q^3)}. \quad (1.19)$$

Some Properties of $l_{p,n}$

In this section we discuss some properties of $l_{p,n}$.

Theorem 2.1 For all positive real numbers p and n, we have

1. $l_{1,1}=1$,
2. $l_{p,1}=1$,
3. $l_{p,n}\cdot = l_{p,1/n}$,
4. $l_{p,n}\cdot = l_{n,p}$.

Proof of (i): It is obvious.

Proof of (ii): For q=$e^{-\pi\sqrt{1/p}}$, we have

$$l_{p,1}=\frac{\psi(-e^{-\pi p\sqrt{1/p}})\varphi(e^{-\pi\sqrt{1/p}})}{e^{-\frac{\pi}{8}(p-1)\sqrt{1/p}}\,\psi(-e^{-\pi\sqrt{1/p}})\varphi(e^{-\pi\, p\sqrt{1/p}})}. \quad (2.1)$$

Employing Lemma 1 and Lemma 2 in (2.1), we complete the proof of (ii).

Proof of (iii): For q=$e^{-\pi\sqrt{n/p}}$, we have

$$l_{p,n}l_{p,\frac{1}{n}} =\frac{\psi(-e^{-\pi p\sqrt{n/p}})\varphi(e^{-\pi\sqrt{n/p}})}{e^{-\frac{\pi}{8}(p-1)\sqrt{\frac{n}{p}}}\psi(-e^{-\pi\sqrt{n/p}})\varphi(e^{-\pi p\sqrt{n/p}})}\times \frac{\psi(-e^{-\pi p\sqrt{1/np}})\varphi(e^{-\pi\sqrt{1/np}})}{e^{-\frac{\pi}{8}(p-1)\sqrt{\frac{1}{np}}}\psi(-e^{-\pi\sqrt{1/np}})\varphi(e^{-\pi p\sqrt{1/np}})}. \quad (2.2)$$

Again Lemma 1 and Lemma 2 in (2.2), we easily obtain the result.

Proof of (iv): Interchanging n and p in (1.9), we have

$$\frac{l_{p,n}}{l_{n,p}}=\frac{\psi(-q^p)\varphi(q)}{q^{\frac{p-1}{8}}\psi(-q)\varphi(q^p)}\frac{q^{\frac{n-1}{8}}\psi(-q)\varphi(q^n)}{\psi(-q^n)\varphi(q)}$$
$$=\frac{e^{(-\pi\sqrt{p/n})^{(n-1)/8}}}{e^{(-\pi\sqrt{n/p})^{(p-1)/8}}}$$
$$\times\frac{\psi(-e^{-\pi p\sqrt{\frac{n}{p}}})\psi(-e^{-\pi\sqrt{\frac{p}{n}}})\varphi(-e^{-\pi n\sqrt{\frac{p}{n}}})\varphi(-e^{-\pi\sqrt{n/p}})}{\psi(-e^{-\pi\sqrt{\frac{n}{p}}})\psi(-e^{-np\sqrt{\frac{p}{n}}})\varphi(-e^{-\pi\sqrt{\frac{p}{n}}})\varphi(-e^{-\pi p\sqrt{n/p}})}.$$

Employing again Lemma 1 and Lemma 2 with some manipulation, we have (iv).

Explicit Values of $l_{p,n}$: We prove general theorems for explicit evaluations of $l_{p,n}$ using Ramanujan's class invariants. Ramanujan's two class invariants G_n and g_n are defined by

$$G_n = 2^{-1/4}\, q^{-1/24}\,\chi(q),$$
$$g_n=2^{-1/4}\, q^{-1/24}\,\chi(-q),\quad q=e^{-\pi\sqrt{n}}. \quad (3.1)$$

where $\chi(q)$ is defined in (1.5) and n is a positive rational number. Using Lemma 5 in (3.1), we obtain

$$G_n=\{4\alpha(1-\alpha)\}^{-1/24},$$
$$g_n=2^{-1/12}(1-\alpha)^{1/12}\alpha^{-1/24}. \quad (3.2)$$

If β has degree d over α, then

$$G_{d^2,n}=\{4\beta(1-\beta)\}^{-1/24},$$
$$g_{d^2,n}=2^{-1/12}(1-\beta)^{1/12}\beta^{-1/24}, \quad (3.3)$$

Theorem 3.1 We have

$$l_{p,n}=q^{(1-p)/4}\left(\frac{G_{n/p}}{G_{np}}\right)^3. \quad (3.4)$$

Proof: By the Lemma 3 and Lemma 4, $l_{p,n}$ becomes

$$l_{p,n}=\frac{1}{q^{(p-1)/4}}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}.$$

By (3.2) and (3.3), we have

$$\{\alpha\,(1-\alpha)\}^{1/8}=\frac{1}{4^{1/8}}\left(\frac{1}{G_{n/p}}\right)^3. \quad (3.5)$$

$$\{\beta\,(1-\beta)\}^{1/8}=\frac{1}{4^{1/8}}\left(\frac{1}{G_{nnp}}\right)^3. \quad (3.6)$$

Employing (3.5) and (3.6), we obtain (3.4).

Remark: One can find the values of $l_{3,23}$, $l_{3,47}$ and $l_{3,71}$ (for (p, n) = (3, 23), (3, 47), (3, 71) in Theorem 3.1 respectively) from the equations [p.227, (5.8)], [p. 229, (5.15)] and [p. 227, (5.25)], respectively of [6].

Corollary 3.4 We have

$$l_{n,n}=q^{(1-n)/4}\,G_{n^2}{}^{-3}. \quad (3.7)$$

Proof: Setting p=n in (3.4), we have

$$l_{n,n}=q^{(1-n)/4}\left(\frac{G_1}{G_{n^2}}\right)^3=q^{(1-n)/4}(G_{n^2})^{-3},\quad \text{as}\quad G_1=1.$$

Remark: Employing the corresponding values of G_n from [6, p.189-193], we obtain $l_{3,3}$, $l_{5,,5}$, $l_{7,7}$ and $l_{9,9}$ for Corollary 3.4.

Explicit Evaluation of Ramanujan's Cubic Continued Fraction: In this section we discuss some formulas of explicit evaluations of Ramanujan's cubic continued fraction G(q) in terms of the product $l_{p,n}$.

Theorem 4.1 We have

$$G^3(-e^{-\pi\sqrt{n/3}})=\frac{1+e^{-2\pi\sqrt{n/3}}l_{3,n}^4-\sqrt{e^{-4\pi\sqrt{n/3}}l_{3,n}^8+34\,e^{-2\pi\sqrt{n/3}}l_{3,n}^4+1}}{16}. \quad (4.1)$$

Proof: Changing q into −q in (1.17) and (1.19), we have

$$1+\frac{1}{G^3(-q)}=-\frac{\psi^4(-q)}{q\psi^4(-q^3)}, \quad (4.2)$$

$$1-8G^3(-q)=\frac{\varphi^4(q)}{\varphi^4(q^3)}. \quad (4.3)$$

Dividing (4.3) by (4.2) and simplifying, we obtain the equation

$$8\,G^6(-q)-(1+q^2l_{3,n}^4)\,G^3(-q)-q^2l_{3,n}^4=0. \quad (4.4)$$

Solving (4.4) for q =$e^{-\pi\sqrt{n/3}}$, we obtain (4.1).

Example: As $l_{3,1}=1$, by Theorem 4.1, we have

$$G(-e^{-\pi\sqrt{1/3}})=\left(\frac{1+e^{-2\pi\sqrt{1/3}}-\sqrt{e^{-4\pi\sqrt{1/3}}+34\,e^{-2\pi\sqrt{1/3}}+1}}{16}\right)^{1/3}.$$

Theorem 4.2:

$$G(e^{-\pi\sqrt{n/9}})=\frac{1+-e^{-2\pi\sqrt{n/9}}\,l_{9,n}-\sqrt{e^{-4\pi\sqrt{n/9}}l_{9,n}^2+10\,e^{-2\pi\sqrt{n/9}}l_{9,n}+1}}{4} \quad (4.5)$$

Proof: Changing q into $-q^2$ in (1.16) and (1.18), we have

$$1+\frac{1}{G(-q^3)}=-\frac{\psi(-q)}{q\,\psi(-q^9)}, \quad (4.6)$$

$$2\,G(-q^2)=1-\frac{\varphi(q)}{\varphi(q^9)}. \quad (4.7)$$

Dividing (4.7) by (4.6) and simplifying, we obtain

$$2G^2(-q^2)-(1+q^2l_{9,n})\,G(-q^2)-q^2l_{9,n}=0 \quad (4.8)$$

Solving (4.8) for q=$e^{-\pi\sqrt{n/9}}$, we have (4.5).

Example: As $l_{9,n}=1$, by Theorem 4.2, we have

$$G(-e^{-\pi/3})=\frac{1+e^{-2\pi/3}-\sqrt{e^{-4\pi/3}+10e^{-2\pi/3}+1}}{4}.$$

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