THIRD ZAGREB INDICES AND COINDICES OF GENERALIZED TRANSFORMATION GRAPHS AND THEIR COMPLEMENTS

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Abstract: In this paper, the expressions for third Zagreb indices and coindices of generalized transformation graphs G^{ab} and their complement graphs $\overline{G^{ab}}$ are obtained.

Keywords: Generalized transformation graphs $\boldsymbol{G^{ab}}$, Zagreb index, Zagreb coindex.

Introduction: Let G be a simple, undirected graph with n vertices and m edges. Let $V(G)$ and $E(G)$ be the vertex set and edge set of G respectively. If u and v are adjacent vertices of G, then the edge connecting them will be denoted by uv. The degree of a vertex u in G is the number of edges incident to it and is denoted by $d_G(u)$. The complement of G, denoted by $\overline{G}$, is a graph having the same vertex set as G, in which two vertices are adjacent if and only if they are not adjacent in G. Thus, the size of $\overline{G}$ is $\binom{n}{2} - m$ and $d_{\overline{G}}(v) = n - 1 - d_G(v)$ holds for all $v \in V(G)$.

For terminology not defined here we refer the reader to [5].

In theoretical chemistry, the physico-chemical properties of chemical compounds are often modeled by means of molecular-graph-based structure-descriptors, which are also referred to as topological indices [4], [8]. The first and the second Zagreb indices, respectively, defined

$M_1(G) = \sum_{u\in V(G)} d_G(u)^2$ and

$M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$

are widely studied degree-based topological indices, that were introduced by Gutman and Trinajstic′ [3] in 1972.

In [2], G. H. Fath-Tabar introduced a new Zagreb index of a graph G named as "third Zagreb index" and is defined as:

$M_3(G) = \sum_{uv\in E(G)} |d_G(u) - d_G(v)|$.

Recently, Veylaki et al. [9] introduced third Zagreb coindex and is defined as:

$\overline{M_3}(G) = \sum_{uv\notin E(G)} |d_G(u) - d_G(v)|$.

The following earlier established results will be needed for the present considerations.

Theorem 1.1 [9] *Let G be a simple graph. Then*

$\overline{M_3}(G) = M_3(\overline{G})$.

Theorem 1.2 [9] *Let G be a simple graph. Then*

$\overline{M_3}(\overline{G}) = M_3(G)$.

Generalized transformation graphs G^ab: The semitotal-point graph $T_2(G)$ of a graph G is a graph whose vertex set is $V(T_2(G)) = V(G) \cup E(G)$ and two vertices are adjacent in $T_2(G)$ if and only if (i) they are adjacent vertices of G or (ii) one is a vertex of G and other is an edge of G incident with it. It was introduced by Sampathkumar and Chikkodimath [7]. Recently some new graphical transformations were defined by Basavanagoud et al. [1], which generalizes the concept of semitotal-point graph.

The generalized transformation graph G^{ab} is a graph whose vertex set is $V(G) \cup E(G)$, and $\alpha, \beta \in V(G^{ab})$. The vertices α and β are adjacent in G^{ab} if and only if $(*)$ and $(**)$ holds: $(*)$ $\alpha, \beta \in V(G)$, α, β are adjacent in G if $a = +$ and α, β are not adjacent in G if $a = -$. $(**)$ $\alpha \in V(G)$ and $\beta \in E(G)$, α, β are incident in G if $b = +$ and α, β are not incident in G if $b = -$.

One can obtain the four graphical transformations of graphs as G^{++}, G^{+-}, G^{-+} and G^{--}. The vertex v_i of G^{ab} corresponding to a vertex v_i of G is referred to as *point vertex* and vertex e_i of G^{ab} corresponding to an edge e_i of G is referred to as *line vertex*.

In [1], we obtained the expressions for first and second Zagreb indices and coindices for generalized transformation graphs G^{ab} and their complements $\overline{G^{ab}}$. Now we obtain the expressions for third Zagreb indices and coindices for generalized transformation graphs G^{ab} and their complements $\overline{G^{ab}}$.

Proposition 2.1 [1] *Let G be a (n,m)-graph. Then the degree of point and line vertices in G^{ab} are*

1. $d_{G^{++}}(v_i) = 2d_G(v_i)$ and $d_{G^{++}}(e_i) = 2$.
2. $d_{G^{+-}}(v_i) = m$ and $d_{G^{+-}}(e_i) = n - 2$.
3. $d_{G^{-+}}(v_i) = n - 1$ and $d_{G^{-+}}(e_i) = 2$.
4. $d_{G^{--}}(v_i) = n + m - 1 - 2d_G(v_i)$ and $d_{G^{--}}(e_i) = n - 2$.

Proposition 2.2 [6] *Let G be a (n,m)-graph. Then the degree of point and line vertices in $\overline{G^{ab}}$ are*

1. $d_{\overline{G^{++}}}(v_i) = n + m - 1 - 2d_G(v_i)$ and $d_{\overline{G^{++}}}(e_i) = n + m - 3$.
2. $d_{\overline{G^{+-}}}(v_i) = n - 1$ and $d_{\overline{G^{+-}}}(e_i) = m + 1$.
3. $d_{\overline{G^{-+}}}(v_i) = m$ and $d_{\overline{G^{-+}}}(e_i) = n + m - 3$.
4. $d_{\overline{G^{--}}}(v_i) = 2d_G(v_i)$ and $d_{\overline{G^{--}}}(e_i) = m + 1$.

Results:

Theorem 3.1 *Let G be a graph with n vertices and m edges. Then* $M_3(G^{++}) \le 2M_3(G) + 4m + 2M_1(G)$.

Proof. Partition the edge set $E(G^{++})$ into subsets E_1 and E_2, where $E_1 = \{uv | uv \in E(G)\}$ and $E_2 = \{ue |\ the\ vertex\ u\ is\ incident\ to\ the\ edge\ e\ in\ G\}$. It is easy to check that $|E_1| = m$ and $|E_2| = 2m$.

$M_3(G^{++}) = \sum_{uv\in E(G^{++})} |d_{G^{++}}(u) - d_{G^{++}}(v)|$

$= \sum_{uv\in E_1} |d_{G^{++}}(u) - d_{G^{++}}(v)| + \sum_{ue\in E_2} |d_{G^{++}}(u) - d_{G^{++}}(e)|$

By Proposition 2.1, we have

$=\sum_{uv\in E_1}|2d_G(u)-2d_G(v)|+\sum_{ue\in E_2}|2-2d_G(u)|$

$\leq 2M_3(G)+\sum_{u\in V(G)}d_G(u)(|2|+|2d_G(u)|)$

$M_3(G^{++})\leq 2M_3(G)+4m+2M_1(G)$.

Theorem 3.2 *Let G be a graph with n vertices and m edges. Then*

$M_3(\overline{G^{++}})\leq 2\overline{M_3}(G)+2m(n-2)+4m^2-2M_1(G)$.

Proof. Partition the edge set $E(\overline{G^{++}})$ into subsets E_1, E_2 and E_3, where

$E_1=\{uv|uv\notin E(G)\}$, $E_2=\{ue|the\ vertex\ u\ is\ not\ incident\ to\ the\ edge\ e\ in\ G\}$ and $E_3=\{ef|e,f\in E(G)\}$. It is easy to check that $|E_1|=\binom{n}{2}-m$, $|E_2|=m(n-2)$ and $|E_3|=\binom{m}{2}$.

$M_3(\overline{G^{++}})=\sum_{uv\in E(\overline{G^{++}})}|d_{\overline{G^{++}}}(u)-d_{\overline{G^{++}}}(v)|$

$=\sum_{uv\in E_1}|d_{\overline{G^{++}}}(u)-d_{\overline{G^{++}}}(v)|+\sum_{ue\in E_2}|d_{\overline{G^{++}}}(u)-d_{\overline{G^{++}}}(e)|+\sum_{ef\in E_3}|d_{\overline{G^{++}}}(e)-d_{\overline{G^{++}}}(f)|$

By Proposition 2.2, we have

$=\sum_{uv\notin E(G)}|n+m-1-2d_G(u)-(n+m-1-2d_G(v))|+\sum_{ue\in E_2}|n+m-1-2d_G(u)-n-m+3|+\sum_{ef\in E_3}|n+m-3-n-m+3|$

$=\sum_{uv\notin E(G)}|-2d_G(u)+2d_G(v)|+\sum_{ue\in E_2}|2-2d_G(u)|$

$=2\overline{M_3}(G)+\sum_{u\in V(G)}(m-d_G(u))(|2-2d_G(u)|)$

$\leq 2\overline{M_3}(G)+\sum_{u\in V(G)}(m-d_G(u))(|2|+|2d_G(u)|)$

$M_3(\overline{G^{++}})\leq 2\overline{M_3}(G)+2m(n-2)+4m^2-2M_1(G)$.

Theorem 3.3 *Let G be a graph with n vertices and m edges. Then*

$\overline{M_3}(G^{++})\leq 2\overline{M_3}(G)+2m(n-2)+4m^2-2M_1(G)$.

Proof. The proof of the theorem follows from Theorem 1.1 and Theorem 3.2.

Theorem 3.4 *Let G be a graph with n vertices and m edges. Then* $\overline{M_3}(\overline{G^{++}})\leq 2M_3(G)+4m+2M_1(G)$.

Proof. The proof of the theorem follows from Theorem 1.2 and Theorem 3.1.

Theorem 3.5 *Let G be a graph with n vertices and m edges. Then* $M_3(G^{+-})=m(n-2)|m-n+2|$.

Proof. Partition the edge set $E(G^{+-})$ into subsets E_1 and E_2, where

$E_1=\{uv|uv\in E(G)\}$ and $E_2=\{ue|the\ vertex\ u\ is\ not\ incident\ to\ the\ edge\ e\ in\ G\}$. It is easy to check that $|E_1|=m$ and $|E_2|=m(n-2)$.

$M_3(G^{+-})=\sum_{uv\in E(G^{+-})}|d_{G^{+-}}(u)-d_{G^{+-}}(v)|$

$=\sum_{uv\in E_1}|d_{G^{+-}}(u)-d_{G^{+-}}(v)|+\sum_{ue\in E_2}|d_{G^{+-}}(u)-d_{G^{+-}}(e)|$

In view of Proposition 2.1, we have

$=\sum_{uv\in E_1}|m-m|+\sum_{ue\in E_2}|m-(n-2)|$

$M_3(G^{+-})=m(n-2)|m-n+2|$.

Theorem 3.6 *Let G be a graph with n vertices and m edges. Then* $M_3(\overline{G^{+-}})=2m|n-m-2|$.

Proof. Partition the edge set $E(\overline{G^{+-}})$ into subsets E_1, E_2 and E_3, where

$E_1=\{uv|uv\notin E(G)\}$, $E_2=\{ue|the\ vertex\ u\ is\ incident\ to\ the\ edge\ e\ in\ G\}$ and $E_3=\{ef|e,f\in E(G)\}$. It is easy to check that $|E_1|=\binom{n}{2}-m$, $|E_2|=2m$ and $|E_3|=\binom{m}{2}$.

$M_3(\overline{G^{+-}})=\sum_{uv\in E(\overline{G^{+-}})}|d_{\overline{G^{+-}}}(u)-d_{\overline{G^{+-}}}(v)|$

$=\sum_{uv\in E_1}|d_{\overline{G^{+-}}}(u)-d_{\overline{G^{+-}}}(v)|+\sum_{ue\in E_2}|d_{\overline{G^{+-}}}(u)-d_{\overline{G^{+-}}}(e)|+\sum_{ef\in E_3}|d_{\overline{G^{+-}}}(e)-d_{\overline{G^{+-}}}(f)|$

In view of Proposition 2.2, we have

$=\sum_{uv\notin E(G)}|n-1-(n-1)|+\sum_{ue\in E_2}|n-1-(m+1)|+\sum_{ef\in E_3}|m+1-(m+1)|$

$=\sum_{ue\in E_2}|n-m-2|$

$M_3(\overline{G^{+-}})=2m|n-m-2|$.

Theorem 3.7 *Let G be a graph with n vertices and m edges. Then* $\overline{M_3}(G^{+-})=2m|n-m-2|$.

Proof. The proof of the theorem follows from Theorem 1.1 and Theorem 3.6.

Theorem 3.8 *Let G be a graph with n vertices and m edges. Then* $\overline{M_3}(\overline{G^{+-}})=m(n-2)|m-n+2|$.

Proof. The proof of the theorem follows from Theorem 1.2 and Theorem 3.5.

Theorem 3.9 *Let G be a graph with n vertices and m edges. Then* $M_3(G^{-+})=2m|n-3|$.

Proof. Partition the edge set $E(G^{-+})$ into subsets E_1 and E_2, where

$E_1=\{uv|uv\notin E(G)\}$ and $E_2=\{ue|the$ vertex $u\ is\ incident\ to\ the\ edge\ e\ in\ G\}$. It is easy to check that $|E_1|=\binom{n}{2}-m$ and $|E_2|=2m$.

$M_3(G^{-+})=\sum_{uv\in E(G^{-+})}|d_{G^{-+}}(u)-d_{G^{-+}}(v)|$

$=\sum_{uv\in E_1}|d_{G^{-+}}(u)-d_{G^{-+}}(v)|+\sum_{ue\in E_2}|d_{G^{-+}}(u)-d_{G^{-+}}(e)|$

From Proposition 2.1, we have

$=\sum_{uv\in E_1}|n-1-(n-1)|+\sum_{ue\in E_2}|n-1-2|$

$M_3(G^{-+})=2m|n-3|$.

Theorem 3.10 *Let G be a graph with n vertices and m edges. Then* $M_3(\overline{G^{-+}})=m(n-2)|3-n|$.

Proof. Partition the edge set $E(\overline{G^{-+}})$ into subsets E_1, E_2 and E_3, where

$E_1=\{uv|uv\in E(G)\}$, $E_2=\{ue|the\ vertex\ u\ is\ not\ incident\ to\ the\ edge\ e\ in\ G\}$ and $E_3=\{ef|e,f\in E(G)\}$. It is easy to check that $|E_1|=m$, $|E_2|=m(n-2)$ and $|E_3|=\binom{m}{2}$.

$M_3(\overline{G^{-+}})=\sum_{uv\in E(\overline{G^{-+}})}|d_{\overline{G^{-+}}}(u)-d_{\overline{G^{-+}}}(v)|$

$=\sum_{uv\in E_1}|d_{\overline{G^{-+}}}(u)-d_{\overline{G^{-+}}}(v)|+\sum_{ue\in E_2}|d_{\overline{G^{-+}}}(u)-d_{\overline{G^{-+}}}(e)|+\sum_{ef\in E_3}|d_{\overline{G^{-+}}}(e)-d_{\overline{G^{-+}}}(f)|$

From Proposition 2.2, we have

$=\sum_{uv\in E_1} |m-m| + \sum_{ue\in E_2} |m-(n+m-3)| + \sum_{ef\in E_3} |n+m-3-(n+m-3)| = \sum_{ue\in E_2} |3-n|$

$M_3(\overline{G^{-+}}) = m(n-2)|3-n|$.

Theorem 3.11 *Let G be a graph with n vertices and m edges.* Then $\overline{M_3}(G^{-+}) = m(n-2)|3-n|$.

Proof. The proof of the theorem follows from Theorem 1.1 and Theorem 3.10.

Theorem 3.12 *Let G be a graph with n vertices and m edges. Then* $\overline{M_3}(\overline{G^{-+}}) = 2m|n-3|$.

Proof. The proof of the theorem follows from Theorem 1.2 and Theorem 3.9.

Theorem 3.13 *Let G be a graph with n vertices and m edges. Then*

$M_3(G^{--}) \leq 2\overline{M_3}(G) + m(m+1)(n-2) + 4m^2 - 2M_1(G)$.

Proof. Partition the edge set $E(G^{--})$ into subsets E_1 and E_2, where $E_1 = \{uv | uv \notin E(G)\}$ and $E_2 = \{ue | the\ vertex\ u\ is\ not\ incident\ to\ the\ edge\ e\ in\ G\}$.It is easy to check that $|E_1| = \binom{n}{2} - m$ and $|E_2| = m(n-2)$.

$M_3(G^{--}) = \sum_{uv\in E(G^{--})} |d_{G^{--}}(u) - d_{G^{--}}(v)|$

$=\sum_{uv\in E_1} |d_{G^{--}}(u) - d_{G^{--}}(v)| + \sum_{ue\in E_2} |d_{G^{--}}(u) - d_{G^{--}}(e)|$

In view of Proposition 2.1, we have

$=\sum_{uv\in E_1} |n+m-1-2d_G(u) - (n+m-1-2d_G(v))| + \sum_{ue\in E_2} |n+m-1-2d_G(u) - (n-2)|$

$\leq 2\overline{M_3}(G) + \sum_{u\in V(G)} (m - d_G(u))(|m+1| + |2d_G(u)|)$

$M_3(G^{--}) \leq 2\overline{M_3}(G) + m(m+1)(n-2) + 4\,m^2 - 2M_1(G)$.

Theorem 3.14 *Let G be a graph with n vertices and m edges. Then*

$M_3(\overline{G^{--}}) \leq 2M_3(G) + 2M_1(G) + (m+1)2m$.

Proof. Partition the edge set $E(\overline{G^{--}})$ into subsets E_1, E_2 and E_3, where $E_1 = \{uv | uv \in E(G)\}$, $E_2 = \{ue | the\ vertex\ u\ is\ incident\ to\ the\ edge\ e\ in\ G\}$ and $E_3 = \{ef | e, f \in E(G)\}$. It is easy to check that $|E_1| = m$, $|E_2| = 2m$ and $|E_3| = \binom{m}{2}$.

$M_3(\overline{G^{--}}) = \sum_{uv\in E(\overline{G^{--}})} |d_{\overline{G^{--}}}(u) - d_{\overline{G^{--}}}(v)|$

$=\sum_{uv\in E_1} |d_{\overline{G^{--}}}(u) - d_{\overline{G^{--}}}(v)| + \sum_{ue\in E_2} |d_{\overline{G^{--}}}(u) - d_{\overline{G^{--}}}(e)| + \sum_{ef\in E_3} |d_{\overline{G^{--}}}(e) - d_{\overline{G^{--}}}(f)|$

In view of Proposition 2.2, we have

$=\sum_{uv\in E_1} |2d_G(u) - 2d_G(v)| + \sum_{ue\in E_2} |2d_G(u) - (m+1)| + \sum_{ef\in E_3} |m+1-(m+1)|$

$=2M_3(G) + \sum_{u\in V(G)} d_G(u)(|2d_G(u) - (m+1)|)$

$M_3(\overline{G^{--}}) \leq 2M_3(G) + 2M_1(G) + (m+1)2m$.

Theorem 3.15 *Let G be a graph with n vertices and m edges. Then*

$\overline{M_3}(G^{--}) \leq 2M_3(G) + 2M_1(G) + (m+1)2m$.

Proof. The proof of the theorem follows from Theorem 1.1 and Theorem 3.14.

Theorem 3.16 *Let G be a graph with n vertices and m edges. Then*

$\overline{M_3}(\overline{G^{--}}) \leq 2\overline{M_3}(G) + m(m+1)(n-2) + 4m^2 - 2M_1(G)$.

Proof. The proof of the theorem follows from Theorem 1.2 and Theorem 3.13.

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