EMT LABELINGS AND SEMT LABELINGS OF CONNECTED UNICYCLIC *(p,q)* GRAPHS WITH MAGIC CONSTANTS *p+q+3* AND *3p*

A.SHAKILA JEMIMA, K.KAYATHRI

Abstract: A graph that admits an edge-magic (super edge-magic) total labeling is called a EMT graph (SEMT graph). In this paper, we enumerate the SEMT labelings and EMT labelings with magic constants p+q+3 and 3p, of connected unicyclic graphs, that contain a star of maximum size m, where m = p - 1, p - 2.

Keywords: Super edge-magic, unicyclic graphs

Introduction: The edge-magic concept was first introduced and studied by Kotzig and Rosa [5], under a different name, 'the magic valuation'. Let G be a simple, connected and undirected graph with p vertices and q edges. An edge-magic total labeling f is a bijection from $V(G) \cup E(G)$ to the set of integers $\{1, 2, \ldots, p+q\}$ such that if xy is an edge of G, then $f(x)+f(y)+f(xy)=\lambda$ for some integer constant λ. A graph that admits an edge-magic total labeling is called a EMT graph. The super edge-magic notion was first introduced by Enomota, Llado, Nakamigawa and Ringel [2]. A super edge-magic total labeling is an edge-magic total labeling f with $f(V(G)) = \{1,2,\ldots p\}$ and $f(E(G)) = \{p+1, p+2, \ldots, p+q\}$. A graph that admits a super edge-magic total labeling is called a SEMT graph.

Though the EMT labelings of several classes of graphs have been studied by many authors[3], not much work is done on enumeration of EMT labelings or SEMT labelings.

In [4], we enumerated SEMT labelings of connected unicyclic SEMT graphs with magic constant $\lambda = p + q + 3$ that contain a star of maximum size m, with $p - 4 \leq m \leq p - 1$. In this paper, we enumerate EMT labelings of such graphs. Using duality, we enumerate the SEMT labelings and EMT labelings with magic constant $\lambda = 3p$, of these graphs.

Connected unicyclic graphs with the magic constant *p+q+3*:

Notations:

1. Let $\mathcal{G}$ denote the class of all connected unicyclic SEMT graphs with $\lambda = p+q+3$.
2. Let G be any graph in $\mathcal{G}$, with a SEMT labeling f with $\lambda = p + q + 3$. For our convenience, we fix $V(G)$ and $E(G)$ as follows:
   $V(G) = \{v_1, v_2, \ldots, v_p\}$, $E(G)=\{e_1, e_2, \ldots, e_q\}$,
   where $f(v_i)=i$ and $f(e_i) = p + q + 1 - i$.
3. Let $\mathcal{G}_m$ denote the class of all connected unicyclic SEMT graphs that contain a star of maximum size m, and magic constant $\lambda = p + q + 3$.

Observations: Let G be any graph in $\mathcal{G}$. Let f be an SEMT labeling of G with the magic constant p+q+3. Then

1. vp is not adjacent with any v_i for $3 \leq i \leq p$.
2. Sum of the labels of the vertices incident with the edge e_i is $i+2$ for $1 \leq i \leq p$.
3. $e_1 = v_1v_2$ and $e_2 = v_1v_3$.
4. $e_q (= e_p)$ is not incident with v_1.

Known Results:

Result 1 [3] *Let G be any graph in $\mathcal{G}$. Then G contains K_3 (and so $K_3 = C_3$ is the only cycle in G). Moreover, in any SEMT labeling with $\lambda = p + q + 3$, the vertex v_1 lies in K_3.*

Result 2 [3] *The graph $K_{1,p-1} + e$ with $p \geq 4$, has $\left\lfloor \frac{p+1}{2} \right\rfloor$ SEMT labelings with $\lambda = p + q + 3$.*

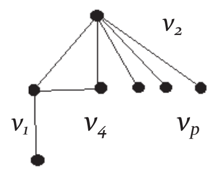
Result 3[3] *When $p \geq 6$, the total number of SEMT labelings of graphs in $\mathcal{G}_{p-2}$ with $\lambda = p + q + 3$ is*

$$S = \begin{cases} \frac{(p-3)(p-2)+6}{4} \ if \ p \equiv 0 \ (mod\ 4) \\ \frac{(p-3)(p-2)+4}{4} if \ p \equiv 2 \ (mod\ 4) \\ \frac{(p-3)(p-1)+4}{4} \ if \ p \equiv 1 \ or \ 3 \ (mod\ 4). \end{cases}$$

In Result 3, we have completely studied the structure of the graphs in $\mathcal{G}_{p-2}$. Using this structure only, the enumeration is carried out. For our future reference, the sturcture mentioned in the proof of Result 3, is stated as Result 4.

Result 4 *Let G be a graph in $\mathcal{G}_{p-2}$ with $p \geq 6$. Let f be a SEMT labeling of G with $f(v_i) = i$ and $f(e_i) = p + q + 1 - i$, for $1 \leq i \leq p$ and $\lambda = p + q + 3$. Let v_r be the vertex not in $K_{1,p-2}$.* Then

1. *v_1 and v_2 are the only possible roots of $K_{1,p-2}$.*
2. *When v_2 is the root of $K_{1,p-2}$,*
   2.1 *$v_r = v_3$*
   2.2 *$e_r = v_1v_{r+1} = v_1v_4$*
   2.3 *e_r lies in the cycle*
   2.4 *G is the graph with the SEMT labeling given in Fig. 1.*



v_3 *Fig.1*

3. *When v_1 is the root of $K_{1,p-2}$, then*

3.1. $e_q = e_p = v_r v_{p+2-r}$

3.2. $r \neq \frac{p+2}{2}$

3.3 $4 \leq r \leq p$

3.4. $e_{r-1} = v_k v_{r+1-k}$, *for some k, where* $2 \leq k \leq \left\lfloor \frac{r}{2} \right\rfloor$

3.5. e_{r-1} *lies in the cycle.*

EMT labelings: Theorem 1 *Let G be a SEMT graph with magic constant λ and let G have k pendant vertices. Then every SEMT labeling of G induces* 2^k *EMT labelings with the same magic constant λ.*

Proof Given a SEMT labeling f of G with magic constant λ, by interchanging the label of a pendant vertex of G with the label of its incident edge, we get a new EMT labeling f^* with the same magic constant. Each pendant vertex contributes two EMT labelings under this process. As G has k pendant vertices, there are 2^k EMT labelings with the same magic constant λ.

Theorem 2 *If a graph G has n SEMT labelings and k pendant vertices, then G has* $2^k n$ *EMT labelings.*

Proof Let f_1 and f_2 be two SEMT labelings of a graph G. Let f_1^* be an EMT labeling of G obtained from f_1, by interchanging the labels of some of the pendant vertices with the labels of their incident edges. Similarly, let f_2^* be an EMT labeling of G obtained from f_2, by interchanging the labels of some of the pendant vertices with the labels of their incident edges.

Claim: $f_1^* \neq f_2^*$

Assume the contrary that $f_1^* = f_2^*$. Since $f_1 \neq f_2$, either (i) $f_1(v) \neq f_2(v)$ for some vertex v or (ii) $f_1(e) \neq f_2(e)$ for some edge e.

Suppose (i) holds. Now $f_1^*(v) = f_2^*(v)$ implies that either f_1^* changes the label of v or f_2^* changes the label of v. Without loss of generality, let f_1^* change the label of v. Then v is a pendant vertex and $f_1^*(v) = f_1(x)$ where x is the pendant edge incident with v. So $f_1^*(v) > p$. Now $f_2^*(v) = f_1^*(v) > p$; and so f_2^* also has changed the label of v. Hence $f_1^*(x) = f_1(v) \neq f_2(v) = f_2^*(x)$; and so $f_1^* \neq f_2^*$, a contradiction. Proof is analogous when (ii) holds.

Thus distinct SEMT labelings give rise to distinct EMT labelings by (interchanging the labels of some of the pendant vertices with the labels of their incident edges). Now, using Theorem 1, we get the result.

EMT and SEMT labeling of graphs in $\mathcal{G}_{p-1}$ and $\mathcal{G}_{p-2}$:

Lemma 3 *(i)* $K_{1,p-1}+e$ *is the only graph in* $\mathcal{G}_{p-1}$.

(ii) The two graphs G_1 *and* G_2 *in Fig.2 are the only graphs in* $\mathcal{G}_{p-2}$.

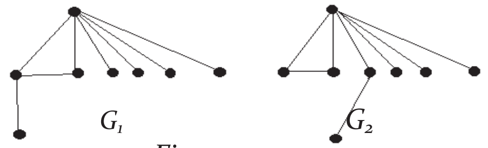


Fig.2

Proof By Result 1, C_3 is the only cycle in graphs in $\mathcal{G}$. Since $\mathcal{G}_{p-1}$ and $\mathcal{G}_{p-2}$ are the subclasses of $\mathcal{G}$, the result follows.

Theorem 4 *The graph* $K_{1,\,p-1} + e$ *with* $p \geq 4$, *has* $2^{p-3}\left\lfloor \frac{p+1}{2} \right\rfloor$ *EMT labelings with λ=p+q+ 3.*

Proof By Theorem 2 and Result 2, the result follows.

Theorem 5 *When* $p \geq 6$,

(i) the total number of SEMT labelings of G_1 *(in Fig.2) with* $\lambda = p + q + 3$ *is*

$$S_1 = \begin{cases} \left\lfloor \frac{p-1}{2} \right\rfloor & if\ p \equiv 0, 3\ (mod\ 6) \\ \left\lceil \frac{p}{2} \right\rceil & otherwise. \end{cases}$$

(ii) the total number of EMT labelings of G_1 *with* $\lambda = p + q + 3$ *is*

$$T_1 = \begin{cases} \left\lfloor \frac{p-1}{2} \right\rfloor 2^{p-3} & if\ p \equiv 0, 3\ (mod\ 6) \\ \left\lceil \frac{p}{2} \right\rceil 2^{p-3} & otherwise. \end{cases}$$

Proof Let v_r be the vertex not in $K_{1,\,p-2}$.

We refer Lemma 3 and Result 4 for the structure of G_1. Now, we have two cases:

Case 1: v_2 is the root of $K_{1,\,p-2}$

By Result 4, G_1 has only one SEMT labeling (Refer Fig.1).

Case 2: v_1 is the root of $K_{1,\,p-2}$ and the neighbour of v_r is incident with e_{r-1}

Now we have two cases:

Case 2.1 $v_{p+2-r} = v_k$

Case 2.2 $v_{p+2-r} = v_{r+1-k}$

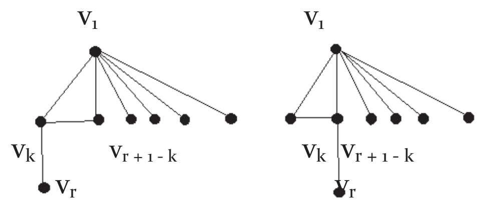


Fig.3

Case 2.1 $v_{p+2-r} = v_k$

Using (3.4) in Result 4, $2 \leq k \leq \left\lfloor \frac{r}{2} \right\rfloor$.

So $2 \leq p+2-r \leq \left\lfloor \frac{r}{2} \right\rfloor$ and $\left\lceil \frac{2p+4}{3} \right\rceil \leq r \leq p$.

As $\frac{p+2}{2} < \left\lceil \frac{2p+4}{3} \right\rceil$, $r \neq \frac{p+2}{2}$; and so (3.2) in Result 4 holds. Counting the number of chances of r, the number of SEMT labelings in Case 2.1 is $S_1' = \left\lfloor \frac{p-1}{3} \right\rfloor$.

Case 2.2 $vp + 2 - r = v\,r + 1 - k$

Then $k = 2r - p - 1$ and $2 \leq 2r - p - 1 \leq \left\lfloor \frac{r}{2} \right\rfloor$. Thus we have $\left\lceil \frac{p+3}{2} \right\rceil \leq r \leq \left\lfloor \frac{2p+2}{3} \right\rfloor$.

Since $\frac{p+2}{2} < \frac{p+3}{2}$, $r \neq \frac{p+2}{2}$; and so (3.2) in Result 4 holds. Counting the number of chances of r, the number of SEMT labelings in Case 2.2 is

$$S_1'' = \begin{cases} \frac{p+1}{6} \; if\ p \equiv 5 \ (mod\ 6) \\ \left\lfloor \frac{p-1}{6} \right\rfloor \quad otherwise. \end{cases}$$

Combining all the cases, the number of SEMT labelings of G_1 is $S_1 = S_1' + S_1'' + 1$ and hence (i) follows. Since G_1 has $p - 3$ pendant vertices, (ii) follows by Theorem 2.

Theorem 6 *When $p \geq 6$, (i) the total number of SEMT labelings of G_2 (in Fig.2) with $\lambda = p + q + 3$ is*

$$= \begin{cases} \frac{p^2-7p+16}{4} \; if\ p \equiv 0\ (mod\ 12) \\ \frac{p^2-6p+5}{4} \; if\ p \equiv 1,5,7,11\ (mod\ 12) \\ \frac{p^2-7p+10}{4} \; if\ p \equiv 2,10\ (mod\ 12) \\ \frac{p^2-6p+9}{4} \; if\ p \equiv 3,9\ (mod\ 12) \\ \frac{p^2-7p+12}{4} \; if\ p \equiv 4,8\ (mod\ 12) \\ \frac{p^2-7p+14}{4} \; if\ p \equiv 6\ (mod\ 12). \end{cases}$$

(ii) the total number of EMT labelings of G_2 with $\lambda = p + q + 3$ is

$$= \begin{cases} \frac{p^2-7p+16}{4} 2^{p-4} \; if\ p \equiv 0\ (mod\ 12) \\ \frac{p^2-6p+5}{4} 2^{p-4} \; if\ p \equiv 1,5,7,11\ (mod\ 12) \\ \frac{p^2-7p+10}{4} 2^{p-4} \; if\ p \equiv 2,10\ (mod\ 12) \\ \frac{p^2-6p+9}{4} 2^{p-4} \; if\ p \equiv 3,9\ (mod\ 12) \\ \frac{p^2-7p+12}{4} 2^{p-4} \; if\ p \equiv 4,8\ (mod\ 12) \\ \frac{p^2-7p+14}{4} 2^{p-4} \; if\ p \equiv 6\ (mod\ 12). \end{cases}$$

Proof Using Lemma 3 and Result 4, v_1 is the only possible root of $K_{1, p-2}$.

Let v_r be the vertex not in $K_{1, p-2}$.

From the structure of G_2, $v_{p+2-r} = v_l$, where $l \neq k$, $r+1-k$.

In Result 3, S gives the total number of SEMT labelings of graphs in $\mathcal{G}_{p-2}$.

In Theorem 5, S_1 gives the total number of SEMT labelings of graph G_1.

Hence the total number of SEMT labelings of G_2 is $S - S_1$ and (i) follows.

Since G_2 has p - 4 pendant vertices, (ii) follows by Theorem 2.

Theorem 7 *When $p \geq 6$, the total number of EMT labelings with $\lambda = p + q + 3$ of graphs in $\mathcal{G}_{p-2}$ is*

$$= \begin{cases} \left(\frac{p-2}{2}\right) 2^{p-3} + \left(\frac{p^2-7p+16}{4}\right) 2^{p-4} \; if\ p \equiv 0(\ mod\ 12) \\ \left(\frac{p+1}{2}\right) 2^{p-3} + \left(\frac{p^2-6p+5}{4}\right) 2^{p-4} \; if\ p \equiv 1,5,7,11\ (mod\ 12) \\ \left(\frac{p}{2}\right) 2^{p-3} + \left(\frac{p^2-7p+10}{4}\right) 2^{p-4} \; if\ p \equiv 2,10\ (mod\ 12) \\ \left(\frac{p-1}{2}\right) 2^{p-3} + \left(\frac{p^2-6p+9}{4}\right) 2^{p-4} \; if\ p \equiv 3,9\ (mod\ 12) \\ \left(\frac{p}{2}\right) 2^{p-3} + \left(\frac{p^2-7p+12}{4}\right) 2^{p-4} \; if\ p \equiv 4,8\ (mod\ 12) \\ \left(\frac{p-2}{2}\right) 2^{p-3} + \left(\frac{p^2-7p+14}{4}\right) 2^{p-4} \; if\ p \equiv 6\ (mod\ 12). \end{cases}$$

Proof By Theorems 2, 5 and 6, the result follows.

SEMT and EMT labelings with λ = 3p:

E.T. Baskoro, I.W. Sudarsana and Y.M. Cholily [1] defined the dual super labeling. If f is a SEMT labeling of a graph G with the magic constant λ, then the dual super labeling f' is defined by $f'(v) = p + 1 - f(v)$, $v \in V(G)$, and $f(e) = 2p + q + 1 - f(e)$, $e \in E(G)$.

Then f' is a SEMT labeling of G with the magic constant $\lambda' = 4p + q + 3 - \lambda$.

It is easy to note that, if f is a SEMT labeling with $\lambda = p + q + 3$, then f' is SEMT labeling with $\lambda' = 3p$, and vice versa.

Applying the duality for Results 2, 3 and Theorems 4, 5 and 6, we get the following results. For completeness, we state these results without proof.

Theorem 8 *The graph $K_{1,\,p-1} + e$ with $p \geq 4$, has $\left\lfloor \frac{p+1}{2} \right\rfloor$ SEMT labelings with $\lambda = 3p$.*

Theorem 9 *When $p \geq 6$, the total number of SEMT labelings of graphs in $\mathcal{G}_{p-2}$ with $\lambda = 3p$ is*

$$S = \begin{cases} \frac{(p-3)(p-2)+6}{4} \; if\ p \equiv 0\ (mod\ 4) \\ \frac{(p-3)(p-2)+4}{4} \; if\ p \equiv 2\ (mod\ 4) \\ \frac{(p-3)(p-1)+4}{4} \; if\ p \equiv 1\ or\ 3\ (mod\ 4). \end{cases}$$

Theorem 10 *When $p \geq 6$, (i) the total number of SEMT labelings of G_1 (in Fig.2) with $\lambda = 3p$ is*

$$S_1 = \begin{cases} \left\lfloor \frac{p-1}{2} \right\rfloor \; if\ p \equiv 0,3\ (mod\ 6) \\ \left\lceil \frac{p}{2} \right\rceil \quad otherwise. \end{cases}$$

(ii) the total number of EMT labelings of G_1 *with $\lambda = 3p$ is*

$$T_1 = = \begin{cases} \left\lfloor \frac{p-1}{2} \right\rfloor 2^{p-3} \; if\ p \equiv 0,3\ (mod\ 6) \\ \left\lceil \frac{p}{2} \right\rceil 2^{p-3} \quad otherwise. \end{cases}$$

Theorem 11 *When p ≥ 6, (i) the total number of SEMT labeling of* G_2 *(in Fig.2) with* $\lambda = 3p$ *is*

$$= \begin{cases} \frac{p^2-7p+16}{4} & if\ p \equiv 0\ (mod\ 12) \\ \frac{p^2-6p+5}{4} & if\ p \equiv 1,5,7,11\ (mod\ 12) \\ \frac{p^2-7p+10}{4} & if\ p \equiv 2,10\ (mod\ 12) \\ \frac{p^2-6p+9}{4} & if\ p \equiv 3,9\ (mod\ 12) \\ \frac{p^2-7p+12}{4} & if\ p \equiv 4,8\ (mod\ 12) \\ \frac{p^2-7p+14}{4} & if\ p \equiv 6\ (mod\ 12). \end{cases}$$

(ii) the total number of EMT labelings of G_2 *with* $\lambda = 3p$ *is*

$$= \begin{cases} \frac{p^2-7p+16}{4} 2^{p-4} & if\ p \equiv 0\ (mod\ 12) \\ \frac{p^2-6p+5}{4} 2^{p-4} & if\ p \equiv 1,5,7,11\ (mod\ 12) \\ \frac{p^2-7p+10}{4} & if\ p \equiv 2,10\ (mod\ 12) \\ \frac{p^2-6p+9}{4} & if\ p \equiv 3,9\ (mod\ 12) \\ \frac{p^2-7p+12}{4} & if\ p \equiv 4,8\ (mod\ 12) \\ \frac{p^2-7p+14}{4} & if\ p \equiv 6\ (mod\ 12). \end{cases}$$

Theorem 12 When p ≥ 6, the total number of EMT labelings with λ = 3p of graphs in $\mathcal{G}_{p-2}$ is

$$= \begin{cases} \left(\frac{p-2}{2}\right) 2^{p-3} + \left(\frac{p^2-7p+16}{4}\right) 2^{p-4} & \text{if } p \equiv 0\ (\text{mod } 12) \\ \left(\frac{p+1}{2}\right) 2^{p-3} + \left(\frac{p^2-6p+5}{4}\right) 2^{p-4} & \text{if } p \equiv 1,5,7,11\ (\text{mod } 12) \\ \left(\frac{p}{2}\right) 2^{p-3} + \left(\frac{p^2-7p+10}{4}\right) 2^{p-4} & \text{if } p \equiv 2,10\ (\text{mod } 12) \\ \left(\frac{p-1}{2}\right) 2^{p-3} + \left(\frac{p^2-6p+9}{4}\right) 2^{p-4} & \text{if } p \equiv 3,9\ (\text{mod } 12) \\ \left(\frac{p}{2}\right) 2^{p-3} + \left(\frac{p^2-7p+12}{4}\right) 2^{p-4} & \text{if } p \equiv 4,8\ (\text{mod } 12) \\ \left(\frac{p-2}{2}\right) 2^{p-3} + \left(\frac{p^2-7p+14}{4}\right) 2^{p-4} & \text{if } p \equiv 6\ (\text{mod } 12). \end{cases}$$

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A. Shakila Jemima, Associate Professor, Department of Mathematics, N.M.S.S.V.N. College, Madurai, Tamil Nadu, India

Dr. K. Kayathri, Associate Professor, PG & Research Department of Mathematics, Thiagarajar College, Madurai, Tamil Nadu, India.