INVERSE IMAGES OF Fs-SUBSETS UNDER AN Fs-FUNCTION – SOME RESULTS

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Abstract: In this paper we prove that inverse of an Fs-function preserves finite Fs-unions and finite Fs-intersections

Keywords: Fs-set, Fs-subset , Fs-function, Image of an Fs-subset, Inverse image of an Fs-set.

Introduction: Ever since Zadeh [18] introduced the notion of fuzzy sets in his pioneering work, several mathematicians studied numerous aspects of fuzzy sets. Recently many researchers put their efforts in order to prove collection of all fuzzy subsets of a given fuzzy set is Boolean algebra under suitable operations [21].Vaddiparthi Yogeswara , G.Srinivas and Biswajit Rath[1] introduced the concept of Fs-set and developed the theory of Fs-sets in order to prove collection of all Fs-subsets of given Fs-set is a complete Boolean algebra under Fs-unions, Fs-intersections and Fs-complements. The Fs-sets they introduced contain Boolean valued membership functions. They are successful in their efforts in proving that result with some conditions. In this paper we prove that inverse of an Fs-function preserves finite Fs-unions and finite Fs-intersections. For smooth reading of the paper, the theory of Fs-sets and Fs-functions in brief is dealt with in first two sections. We denote the largest element of a complete Boolean algebra L_A[1.1] by M_A or 1_A. For all lattice theoretic properties and Boolean algebraic properties one can refer Szasz [13], Garret Birkhoff[14],Steven Givant • Paul Halmos[12] and Thomas Jech[15].For results in topology one can refer[20]

THEORY OF FS-SETS

1.1 Fs-set: Let U be a universal set, $A_1 \subseteq U$ and let $A \subseteq U$ be non-empty. A four tuple $\mathcal{A} = \left(A_1, A, \overline{A}\left(\mu_{1A_1}, \mu_{2A}\right), L_A\right)$ is said be an Fs-set if, and only if

(1) $A \subseteq A_1$

(2) L_A is a complete Boolean Algebra

(3) $\mu_{1A_1}: A_1 \longrightarrow L_A$, $\mu_{2A}: A \longrightarrow L_A$,are functions such that $\mu_{1A_1}|A \geq \mu_{2A}$

(4) $\overline{A}$:A⟶ L_A is defined by $\overline{A}x = \mu_{1A_1}x \wedge \left(\mu_{2A}x\right)^c$, for each $x \in A$

1.2 Fs-subset

Let $\mathcal{A} = \left(A_1, A, \overline{A}\left(\mu_{1A_1}, \mu_{2A}\right), L_A\right)$ and $\mathcal{B} = \left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$ be a pair of Fs-sets. $\mathcal{B}$ is said to be an Fs-subset of $\mathcal{A}$, denoted by $\mathcal{B} \subseteq \mathcal{A}$, if, and only if

(1) $B_1 \subseteq A_1$, $A \subseteq B$

(2) L_B is a complete subalgebra of L_A or $L_B \leq L_A$

(3) $\mu_{1B_1} \leq \mu_{1A_1}|B_1$, and $\mu_{2B}|A \geq \mu_{2A}$

1.3 Proposition: Let $\mathcal{B}$ and $\mathcal{A}$ be a pair of Fs-sets such that $\mathcal{B} \subseteq \mathcal{A}$. Then $\overline{B}x \leq \overline{A}x$ is true for each $x \in A$

1.4 Definition: For some L_X, such that $L_X \leq L_A$ a four tuple $\mathcal{X} = \left(X_1, X, \overline{X}\left(\mu_{1X_1}, \mu_{2X}\right), L_X\right)$ is not an Fs-set if, and only if

(a) $X \nsubseteq X_1$ or (b) $\mu_{1X_1}x \ngeq \mu_{2X}x$, for some $x \in X \cap X_1$

Here onwards, any object of this type is called an Fs-empty set of first kind and we accept that it is an Fs-subset of $\mathcal{B}$ for any $\mathcal{B} \subseteq \mathcal{A}$.

Definition: An Fs-subset $\mathcal{Y}=\left(Y_1, Y, \overline{Y}\left(\mu_{1Y_1}, \mu_{2Y}\right), L_Y\right)$ of $\mathcal{A}$, is said to be an Fs-empty set of second kind if, and only if

(a') $Y_1 = Y = A$ (b') $L_Y \leq L_A$ (c') $\overline{Y} = 0$

1.4.1 Remark: we denote Fs-empty set of first kind or Fs-empty set of second kind by $\Phi_{\mathcal{A}}$and we prove later (1.15), $\Phi_{\mathcal{A}}$ is the least Fs-subset among all Fs-subsets of $\mathcal{A}$.

1.5 Definition: Let $\mathcal{B}_1 = \left(B_{11}, B_1, \overline{B}_1\left(\mu_{1B_{11}}, \mu_{2B_1}\right), L_{B_1}\right)$ and $\mathcal{B}_2 = \left(B_{12}, B_2, \overline{B}_2\left(\mu_{1B_{12}}, \mu_{2B_2}\right), L_{B_2}\right)$ be a pair of Fs-subsets.

(1) We say that $\mathcal{B}_1$and $\mathcal{B}_2$ are (1,5)-equal, if $B_{11} = B_{12}$ and $L_{B_1} = L_{B_2}$

(2) We say that $\mathcal{B}_1$and $\mathcal{B}_2$ are (2,5)-equal, if $B_1 = B_2$ and $L_{B_1} = L_{B_2}$
(3) We say that $\mathcal{B}_1$and $\mathcal{B}_2$ are 3-equal, if $\mathcal{B}_1$and $\mathcal{B}_2$ are (1,5)-equal and $\mu_{1B_{11}} = \mu_{1B_{12}}$
(4) We say that $\mathcal{B}_1$and $\mathcal{B}_2$ are 4-equal, if $\mathcal{B}_1$and $\mathcal{B}_2$ are (2,5)-equal and $\mu_{2B_1} = \mu_{2B_2}$
(5) We say that $\mathcal{B}_1$and $\mathcal{B}_2$ are Total equal denoted $\mathcal{B}_1 = \mathcal{B}_2(T)$, if $\mathcal{B}_1$and $\mathcal{B}_2$ are (2,5)-equal and $\overline{B}_1 = \overline{B}_2$
(6) We say that$\mathcal{B}_1$, $\mathcal{B}_2$ are Full-equal, denoted $\mathcal{B}_1 = \mathcal{B}_2$, if $\mathcal{B}_1$and $\mathcal{B}_2$ are 3-equal and 4-equal.

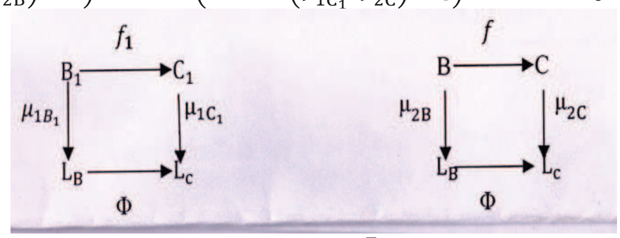
1.6Proposition:
$\mathcal{B}_1 = \left(B_{11}, B_1, \overline{B}_1\left(\mu_{1B_{11}}, \mu_{1B_1}\right), L_{B_1}\right)$ and $\mathcal{B}_2 = \left(B_{12}, B_2, \overline{B}_2\left(\mu_{1B_{12}}, \mu_{2B_2}\right), L_{B_2}\right)$ are Full-equal if, only if $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $\mathcal{B}_2 \subseteq \mathcal{B}_1$.

1.7 **Remark:** Whenever X and Y are Complete Boolean algebra $\Phi \subseteq X \times Y$ be a relation
(a) We say that Φ is (∨,∧)-complete relation on X if, and only if $\vee\Phi(\wedge_{\alpha\in T}\alpha) = \wedge_{\alpha\in T}(\vee\Phi\alpha)$ for any $T \subseteq X$.
(b) We say that Φ is (∨,∨)-complete relation on X if, and only if $\vee\Phi(\vee_{\alpha\in T}\alpha) = \vee_{\alpha\in T}(\vee\Phi\alpha)$ for any $T \subseteq X$
(c) We say that Φ is (∧,∨)-complete relation on X if, and only if $\wedge\Phi(\vee_{\alpha\in T}\alpha) = \vee_{\alpha\in T}(\wedge\Phi\alpha)$ for any $T \subseteq X$
(d) We say that Φ is said to be ∨-increasing on X if, and only if,$\vee\Phi\alpha \leq \vee\Phi\beta$ for any $\alpha, \beta \in X$ such that $\alpha \leq \beta$

1.8 Proposition: Whenever $\Phi: X \longrightarrow Y$ is a complete Boolean algebra homomorphism, then
(1) Φ^{-1} is join increasing on ΦX (2) Φ^{-1} is (∨,∧)-complete relation on ΦX
(3) Φ^{-1} is (∨,∨)-complete relation on ΦX

2.Fs-functions

2.1 Definition: A Triplet (f_1, f, Φ) is said to be is an Fs-Function between two given Fs-subsets $\mathcal{B} = \left(B_1, B, \overline{B}\left(\mu_{1B_1},\mu_{2B}\right), L_B\right)$ and $= \left(C_1, C, \overline{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$ of $\mathcal{A}$, denoted by $(f_1, f, \Phi): \mathcal{B} = \left(B_1, B, \overline{B}\left(\mu_{1B_1},\mu_{2B}\right), L_B\right) \longrightarrow \mathcal{C} = \left(C_1, C, \overline{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$ if, and only if (using the diagrams) .



(Fig-1: Fs-function $\overline{f}: \mathcal{B} \to \mathcal{C}$)

(1a) $f = f_1|_B^C: B \longrightarrow C$ be onto (1b) $\Phi: L_B \longrightarrow L_c$ is complete homomorphism, here onward (f_1, f, Φ) is denoted by $\overline{f}$

IMAGES OF FS-SUBSET

2.2Definition Let $\mathcal{D} \subseteq \mathcal{B}$ and $\overline{f}: \mathcal{B} \longrightarrow \mathcal{C}$ be an Fs-function, where $\mathcal{B} = \left(B_1, B, \overline{B}\left(\mu_{1B_1},\mu_{2B}\right), L_B\right), \mathcal{C} = \left(C_1, C, \overline{C}\left(\mu_{1C_1},\mu_{2C}\right), L_C\right), \mathcal{D} = \left(D_1, D, \overline{D}\left(\mu_{1D_1},\mu_{2D}\right), L_D\right)$, $D = B$ and $f = f_1|_B^C: B \longrightarrow C$ be onto.
Define $\overline{f}(\mathcal{D})$ as follows
$\overline{f}(\mathcal{D}) = \mathcal{E} = \left(E_1, E, \overline{E}\left(\mu_{1E_1}, \mu_{2E}\right) L_E\right)$,
where
(1) $E_1 = f_1(D_1)$
(2) $E = f(D)$
(3) $\mu_{1E_1}: E_1 \longrightarrow L_C$ is defined by $\mu_{1E_1}y = \begin{cases} \mu_{2C}y\vee\left[\mu_{1C_1}y\wedge\left(\vee_{\substack{y=f_1x\\ x\in D_1}}\Phi\mu_{1D_1}x\right)\right], \text{if } y \in C \\ \mu_{1C_1}y\wedge\left(\vee_{\substack{y=f_1x\\ x\in D_1}}\Phi\mu_{1D_1}x\right), \text{if } y \notin C \end{cases}$
(4) $\mu_{2E}: E \longrightarrow L_C$ is defined by $\mu_{2E}y = \mu_{2C}y\vee\left[\mu_{1C_1}y\wedge\left(\vee_{\substack{y=f_1x\\ x\in D}}\Phi\mu_{2D}x\right)\right]$
(5) $L_E = \left(\left[\mu_{1E_1}(E_1)\right]\right)$ =The complete subalgebra generated by $\left[\mu_{1E_1}(E_1)\right]$, where $\left[\mu_{1E_1}(E_1)\right]$ = The complete ideal generated by $\mu_{1E_1}(E_1)$

Inverse Image of an Fs-Subset:

3.1 Definition

Let $\mathcal{D} \subseteq \mathcal{B}$ and $\overline{f}: \mathcal{B} \longrightarrow \mathcal{C}$ be an Fs-function,$\Phi^{-1} \subseteq L_C \times L_B$ be ∨-increasing (Proposition 1.7(d))
$f = f_1|_B^C: B \longrightarrow C$ be onto. Let $\mathcal{E} \subseteq \mathcal{C}$, where $\mathcal{E} = \left(E_1, E, \overline{E}\left(\mu_{1E_1}, \mu_{2E}\right), L_E\right)$

Define $\bar{f}^{-1}(\mathcal{E})$ as follows

$\bar{f}^{-1}(\mathcal{E}) = \mathcal{D} = \left(D_1, D, \bar{D}\left(\mu_{1D_1}, \mu_{2D}\right), L_D\right)$,

where

a) $D_1 = f_1^{-1}(E_1)$

b) $D = f^{-1}(E)$

c) $\mu_{1D_1}: D_1 \to L_D$ is defined by $\mu_{1D_1}x = \begin{cases} \mu_{1B_1}x, \text{whenever } \Phi^{-1}\mu_{1E_1}y = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\vee_{\substack{y=f_1x \\ x\in D_1}} \Phi^{-1}\mu_{1E_1}y\right)\right], x \in B \\ \mu_{1B_1}x \wedge \left(\vee_{\substack{y=f_1x \\ x\in D_1}} \Phi^{-1}\mu_{1E_1}y\right), x \notin B \end{cases}$

d) $\mu_{2D}: D \longrightarrow L_D$ is defined by $\mu_{2D}x = \begin{cases} \mu_{2B}x \quad, \text{whenever } \Phi^{-1}\mu_{2E}y = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\vee_{\substack{y=fx \\ x\in D}} \Phi^{-1}\mu_{2E}y\right)\right] \end{cases}$

e) $L_D = L_B$

3.1.1 Remark: We define,

I. $\Phi_{\mathcal{A}}$ = Fs-empty set of first kind imply $\bar{f}^{-1}(\Phi_{\mathcal{A}})$ = Fs-empty set of first kind.

II. $\Phi_{\mathcal{A}}$ = Fs-empty set of second kind imply $\bar{f}^{-1}(\Phi_{\mathcal{A}})$ = Fs-empty set of second kind.

3.2 Proposition: $\bar{f}^{-1}(\mathcal{E})$ is an Fs-subset of $\mathcal{B}$, if $\Phi^{-1} \subseteq L_C \times L_B$ is $\vee$-increasing

3.3 Proposition: Let $\mathcal{B}$ and $\mathcal{C}$ be any pair of Fs-subsets and $\bar{f}: \mathcal{B} \longrightarrow \mathcal{C}$ be an Fs-function. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Fs-subsets $\mathcal{C}$ such that $\mathcal{E}_1 \subseteq \mathcal{E}_2$ and $E_1 = E_2 = C$, then $\bar{f}^{-1}(\mathcal{E}_1) \subseteq \bar{f}^{-1}(\mathcal{E}_2)$

3.4 Proposition: Let $\mathcal{B}$ and $\mathcal{C}$ be any pair of Fs-subsets and $\bar{f}: \mathcal{B} \longrightarrow \mathcal{C}$ be an Fs-function. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Fs-subsets $\mathcal{C}$ and $E_1 = E_2 = C$, then $\bar{f}^{-1}(\mathcal{E}_1 \cup \mathcal{E}_2)$ and $\bar{f}^{-1}(\mathcal{E}_1) \cup \bar{f}^{-1}(\mathcal{E}_2)$ are full-equal.

Proof: Let $\mathcal{B} = \left(B_1, B, \bar{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right), \mathcal{C} = \left(C_1, C, \bar{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$,

$\mathcal{E}_1 = \left(E_{11}, E_1, \bar{E}_1\left(\mu_{1E_{11}}, \mu_{2E_1}\right), L_{E_1}\right)$ and $\mathcal{E}_2 = \left(E_{12}, E_2, \bar{E}_2\left(\mu_{1E_{12}}, \mu_{2E_2}\right), L_{E_2}\right)$

Let $\mathcal{E}_1 \cup \mathcal{E}_2 = \mathcal{G} = \left(G_1, G, \bar{G}\left(\mu_{1G_1}, \mu_{2G}\right), L_G\right)$, where

(1) $G_1 = E_{11} \cup E_{12}$

(2) $G = E_1 \cap E_2$

(3) $L_G = L_{E_1} \vee L_{E_2}$

(4) $\mu_{1G_1}: G_1 \longrightarrow L_G$ is defined by $\mu_{1G_1}y = \left(\mu_{1E_{11}} \vee \mu_{1E_{12}}\right)y$

(5) $\mu_{2G}: G \longrightarrow L_G$ is defined by $\mu_{2G}x = \left(\mu_{2E_1} \wedge \mu_{2E_2}\right)y = \mu_{2E_1}y \wedge \mu_{2E_2}y$

Suppose $\bar{f}^{-1}(\mathcal{E}_1 \cup \mathcal{E}_2) = \bar{f}^{-1}(\mathcal{G}) = \mathcal{H} = \left(H_1, H, \bar{H}\left(\mu_{1H_1}, \mu_{2H}\right), L_H\right)$, where

(6) $H_1 = f_1^{-1}(G_1) = f_1^{-1}(E_{11} \cup E_{12}) = f_1^{-1}(E_{11}) \cup f_1^{-1}(E_{12})$

(7) $H = f^{-1}(E_1 \cap E_2) = f^{-1}(E_1) \cap f^{-1}(E_2)$

(8) $\mu_{1H_1}: H_1 \to L_H$ is defined by $\mu_{1H_1}x = \begin{cases} \mu_{1B_1}x,, \text{whenever } \Phi^{-1}\mu_{1G_1}y = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\vee_{\substack{\Phi\alpha=\mu_{1G_1}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1G_1}y\right)\right], x \in B \\ \mu_{1B_1}x \wedge \left(\vee_{\substack{\Phi\alpha=\mu_{1G_1}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1G_1}y\right), x \notin B \end{cases}$

(9) $\mu_{2H}: H \longrightarrow L_H$ is defined by $\mu_{2H}x = \begin{cases} \mu_{2B}x \quad, \text{whenever } \Phi^{-1}\mu_{2G}y = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\vee_{\substack{\Phi\alpha=\mu_{2G}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{2G}y\right)\right] \end{cases}$

(10) $L_H = L_B$

Let $\bar{f}^{-1}(\mathcal{E}_1) = \mathcal{D}_1 = \left(D_{11}, D_1, \bar{D}_1\left(\mu_{1D_{11}}, \mu_{2D_1}\right), L_{D_1}\right)$, and $\bar{f}^{-1}(\mathcal{E}_2) = \mathcal{D}_2 = \left(D_{12}, D_2, \bar{D}_2\left(\mu_{1D_{12}}, \mu_{2D_2}\right), L_{D_2}\right)$

where

(11) $D_{11} = f_1^{-1}(E_{11})$

(12) $D_1 = f^{-1}(E_1)$

(13) $\mu_{1D_{11}}: D_{11} \to L_{D_1}$ is defined by $\mu_{1D_{11}}x = \begin{cases} \mu_{1B_1}x, \text{ whenever } \Phi^{-1}\mu_{1E_{11}}y = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right)\right], x \in B \\ \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right), x \notin B \end{cases}$

(14) $\mu_{2D_1}: D_1 \to L_{D_1}$ is defined by $\mu_{2D_1}x = \begin{cases} \mu_{2B}x \quad, \text{whenever } \Phi^{-1}\mu_{2E_1}y = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{2E_1}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{2E_1}y\right)\right] \end{cases}$

(15) $L_{D_1} = L_B$
(16) $D_{12} = f_1^{-1}(E_{12})$
(17) $D_2 = f^{-1}(E_2)$

(18) $\mu_{1D_{12}}: D_{12} \to L_{D_2}$ is defined by $\mu_{1D_{12}}x = \begin{cases} \mu_{1B_1}x, \text{ whenever } \Phi^{-1}\mu_{1E_{12}}y = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)\right], x \in B \\ \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right), x \notin B \end{cases}$

(19) $\mu_{2D_2}: D_2 \to L_{D_2}$ is defined by $\mu_{2D_2}x = \begin{cases} \mu_{2B}x \quad, \text{whenever } \Phi^{-1}\mu_{2E_2}y = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{2E}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{2E_2}y\right)\right] \end{cases}$

(20) $L_{D_2} = L_B$
Suppose $\bar{f}^{-1}(\mathcal{E}_1) \cup \bar{f}^{-1}(\mathcal{E}_2) = \mathcal{D}_1 \cup \mathcal{D}_2 = \mathcal{F} = \left(F_1, F, \bar{F}\left(\mu_{1F_1,}\mu_{2F}\right), L_F\right)$, where
(21) $F_1 = D_{11} \cup D_{12} = f_1^{-1}(E_{11}) \cup f_1^{-1}(E_{12})$
(22) $F = D_1 \cap D_2 = f^{-1}(E_1) \cap f^{-1}(E_2)$
(23) $L_F = L_{D_1} \vee L_{D_2} = L_B$
(24) $\mu_{1F_1}: F_1 \longrightarrow L_F$ is defined by $\mu_{1F_1}x = \left(\mu_{1D_{11}} \vee \mu_{1D_{12}}\right)x$
(25) $\mu_{2F}: F \longrightarrow L_F$ is defined by $\mu_{2F}x = \left(\mu_{2D_1} \wedge \mu_{2D_2}\right)x$
Need to show that, $\mathcal{H}$ and $\mathcal{F}$ are full-equal i.e.
(26) $H_1 = F_1, H = F$
(27) $L_H = L_F$
(28) $\mu_{1H_1}x = \mu_{1F_1}x$, $\mu_{2H}x = \mu_{2F}x$
(26) follows from (6), (7), (21) and (22)
(27) follows from (8) and (23)
Sufficient to show that, $\mu_{1H_1}x = \mu_{1F_1}x$, $\mu_{2H}x = \mu_{2F}x$

$\mu_{1F_1}x = \left(\mu_{1D_{11}} \vee \mu_{1D_{12}}\right)x = \mu_{1D_{11}}x \vee \mu_{1D_{12}}x$ for $x \in B$

$= \left\{\mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right)\right]\right\} \vee \left\{\mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)\right]\right\}$

$= \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left\{\left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right) \vee \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)\right\}\right]$

$\mu_{1H_1}x = \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1G_1}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1G_1}y\right)\right]$

$= \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1G_1}y \\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{1E_{11}} \vee \mu_{1E_{12}}\right)y\right)\right]$

$= \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1G_1}y \\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{1E_{11}}y \vee \mu_{1E_{12}}y\right)\right)\right]$

Case (I): For $x \in B, y \in E_{11} \cap E_{12}$

$\bigvee_{\substack{\Phi\alpha=\mu_{1G_1}y\\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{1E_{11}}y \vee \mu_{1E_{12}}y\right) = \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right) \vee \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)$

$\Phi^{-1}$is $(\vee,\vee)$-complete relation on L_C if, and only if $\vee\Phi^{-1}\left(\vee_{i\epsilon I}\beta_i\right) = \vee\left(\vee_{i\epsilon I}\Phi^{-1}\beta_i\right)$

$\Rightarrow \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1G_1}y\\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{1E_{11}}y \vee \mu_{1E_{12}}y\right)\right) = \mu_{1B_1}x \wedge \left[\left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right) \vee \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)\right]$

$\Rightarrow \mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left(\bigvee_{\substack{\Phi\alpha=\mu_{1G_1}y\\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{1E_{11}}y \vee \mu_{1E_{12}}y\right)\right)\right]$

$= \mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left\{\left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right) \vee \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)\right\}\right]$

$\Rightarrow \mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left(\bigvee_{\substack{\Phi\alpha=\mu_{1G_1}y\\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{1E_{11}} \vee \mu_{1E_{12}}\right)y\right)\right]$

$= \mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left\{\left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right) \vee \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)\right\}\right]$

$\Rightarrow \mu_{1H_1}x = \mu_{1F_1}x$

Case (II): $x \in B, y \in E_{11}, y \notin E_{12}$

$\mu_{1H_1}x = \mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right)\right] = \mu_{1F_1}x$

Case (III): $x \in B, y \in E_{12}, y \notin E_{11}$

$\mu_{1H_1}x = \mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)\right] = \mu_{1F_1}x$

Case (IV): (a) For $x \notin B, y \in E_{11} \cap E_{12}$

$\bigvee_{\substack{\Phi\alpha=\mu_{1G_1}y\\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{1E_{11}}y \vee \mu_{1E_{12}}y\right) = \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right) \vee \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)$

(∵prop 1.7 (c))

$\Rightarrow \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1G_1}y\\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{1E_{11}}y \vee \mu_{1E_{12}}y\right)\right)$

$= \mu_{1B_1}x \wedge \left[\left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right) \vee \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)\right]$

$\Rightarrow \mu_{1H_1}x = \mu_{1F_1}x$

Case (V): $x \notin B, y \in E_{11}, y \notin E_{12}$

$\mu_{1H_1}x = \mu_{1B_1}x\wedge\left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right) = \mu_{1F_1}x$

Case (VI): $x \notin B, y \in E_{12}, y \notin E_{11}$

$\mu_{1H_1}x = \mu_{1B_1}x\wedge\left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right) = \mu_{1F_1}x$

$\mu_{2H}x = \mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left(\bigvee_{\substack{\Phi\alpha=\mu_{2G}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{2G}y\right)\right]$

$= \mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left(\bigvee_{\substack{\Phi\alpha=\mu_{2G}y\\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{2E_1}y \wedge \mu_{2E_2}y\right)\right)\right]$

$\mu_{2F}x = \left(\mu_{2D_1} \wedge \mu_{2D_2}\right)x = \mu_{2D_1}x \wedge \mu_{2D_2}x$

$= \left[\mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left(\bigvee_{\substack{\Phi\alpha=\mu_{2E_1}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{2E_1}y\right)\right]\right]$

$$\wedge\left[\mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left(\vee_{\substack{\Phi\alpha=\mu_{2E_2}y\\ \alpha\in L_B}}\Phi^{-1}\mu_{2E_2}y\right)\right]\right]$$

$$=\mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left[\begin{array}{c}\left(\vee_{\substack{\Phi\alpha=\mu_{2E_1}y\\ \alpha\in L_B}}\Phi^{-1}\mu_{2E_1}y\right)\\ \wedge\left(\vee_{\substack{\Phi\alpha=\mu_{2E_2}y\\ \alpha\in L_B}}\Phi^{-1}\mu_{2E_2}y\right)\end{array}\right]\right]$$

Case (VII): For $x\in B, y\in E_1\cap E_2$

$$\vee_{\substack{\Phi\alpha=\mu_{2G}y\\ \alpha\in L_B}}\Phi^{-1}\left(\mu_{2E_1}y\wedge\mu_{2E_2}y\right)=\left(\vee_{\substack{\Phi\alpha=\mu_{2E_1}y\\ \alpha\in L_B}}\Phi^{-1}\mu_{2E_1}y\right)\wedge\left(\vee_{\substack{\Phi\alpha=\mu_{2E_2}y\\ \alpha\in L_B}}\Phi^{-1}\mu_{2E_2}y\right)\ (\because \text{prop 1.7 (a)})$$

$$\Rightarrow \mu_{1B_1}x\wedge\left(\vee_{\substack{\Phi\alpha=\mu_{2G}y\\ \alpha\in L_B}}\Phi^{-1}\left(\mu_{2E_1}y\wedge\mu_{2E_2}y\right)\right)$$

$$=\mu_{1B_1}x\wedge\left[\left(\vee_{\substack{\Phi\alpha=\mu_{2E_1}y\\ \alpha\in L_B}}\Phi^{-1}\mu_{2E_1}y\right)\wedge\left(\vee_{\substack{\Phi\alpha=\mu_{2E_2}y\\ \alpha\in L_B}}\Phi^{-1}\mu_{2E_2}y\right)\right]$$

$$\Rightarrow \mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left(\vee_{\substack{\Phi\alpha=\mu_{2G}y\\ \alpha\in L_B}}\Phi^{-1}\left(\mu_{2E_1}y\wedge\mu_{2E_2}y\right)\right)\right]=\mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left[\left(\vee_{\substack{\Phi\alpha=\mu_{2E_1}y\\ \alpha\in L_B}}\Phi^{-1}\mu_{2E_1}y\right)\wedge\left(\vee_{\substack{\Phi\alpha=\mu_{2E_1}y\\ \alpha\in L_B}}\Phi^{-1}\mu_{2E_2}y\right)\right]\right]$$

$$\Rightarrow \mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left(\vee_{\substack{y=fx\\ x\in H}}\Phi^{-1}\left(\mu_{2E_1}y\wedge\mu_{2E_2}y\right)\right)\right]$$

$$=\mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left[\begin{array}{c}\left(\vee_{\substack{\Phi\alpha=\mu_{2E_1}y\\ \alpha\in L_B}}\Phi^{-1}\mu_{2E_1}y\right)\\ \wedge\left(\vee_{\substack{\Phi\alpha=\mu_{2E_2}y\\ \alpha\in L_B}}\Phi^{-1}\mu_{2E_2}y\right)\end{array}\right]\right]$$

$\mu_{2H}x=\mu_{2F}x$

Case (VIII): $x\in B, y\in E_1, y\notin E_2$

$$\mu_{2H}x=\mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left(\vee_{\substack{\Phi\alpha=\mu_{2E_1}y\\ \alpha\in L_B}}\Phi^{-1}\mu_{2E_1}y\right)\right]=\mu_{2F}x$$

Case (IX): $x\in B, y\in E_2, y\notin E_1$

$$\mu_{2H}x=\mu_{2B}x\vee\left[\mu_{1B_1}x\wedge\left(\vee_{\substack{\Phi\alpha=\mu_{2E_2}y\\ \alpha\in L_B}}\Phi^{-1}\mu_{2E_2}y\right)\right]=\mu_{2F}x$$

Hence $\overline{f}^{-1}(\mathcal{E}_1\cup\mathcal{E}_2)$ and $\overline{f}^{-1}(\mathcal{E}_1)\cup\overline{f}^{-1}(\mathcal{E}_2)$ are full-equal.

3.5 Proposition: Let $\mathcal{B}$ and $\mathcal{C}$ be any pair of Fs-subsets and $\bar{f}:\mathcal{B}\to\mathcal{C}$ be an Fs-function. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Fs-subsets $\mathcal{C}$ and $E_1=E_2=C$, then $\overline{f}^{-1}(\mathcal{E}_1\cap\mathcal{E}_2)$ and $\overline{f}^{-1}(\mathcal{E}_1)\cap\overline{f}^{-1}(\mathcal{E}_2)$ are full-equal.

Proof: Let $\mathcal{B}=\left(B_1,B,\overline{B}\left(\mu_{1B_1},\mu_{2B}\right),L_B\right)$, $\mathcal{C}=\left(C_1,C,\overline{C}\left(\mu_{1C_1},\mu_{2C}\right),L_C\right)$,

$\mathcal{E}_1=\left(E_{11},E_1,\overline{E}_1\left(\mu_{1E_{11}},\mu_{2E_1}\right),L_{E_1}\right)$ and $\mathcal{E}_2=\left(E_{12},E_2,\overline{E}_2\left(\mu_{1E_{12}},\mu_{2E_2}\right),L_{E_2}\right)$

Let $\mathcal{E}_1=\left(E_{11},E_1,\overline{E}_1\left(\mu_{1E_{11}},\mu_{2E_1}\right),L_{E_1}\right)$ and $\mathcal{E}_2=\left(E_{12},E_2,\overline{E}_2\left(\mu_{1E_{12}},\mu_{2E_2}\right),L_{E_2}\right)$

Let $\mathcal{E}_1\cap\mathcal{E}_2=\mathcal{F}=\left(F_1,F,\overline{F}\left(\mu_{1F_1},\mu_{2F}\right),L_F\right)$, where

(1) $F_1=E_{11}\cap E_{12}$, $F=E_1\cup E_2$

(2) $L_F=L_{E_1}\wedge L_{E_2}$

(3) $\mu_{1F_1}:F_1\longrightarrow L_F$ is defined by $\mu_{1F}y=\left(\mu_{1E_{11}}\wedge\mu_{1E_{12}}\right)y=\mu_{1E_{11}}y\wedge\mu_{1E_{12}}y$

(4) $\mu_{2F}:F\longrightarrow L_F$ is defined by $\mu_{2F}y=\left(\mu_{2E_1}\vee\mu_{2E_2}\right)$

Suppose $\overline{f}^{-1}(\mathcal{E}_1\cap\mathcal{E}_2)=\overline{f}^{-1}(\mathcal{F})=\mathcal{G}=\left(G_1,G,G\left(\mu_{1G_1},\mu_{2G}\right),L_G\right)$, where

(5) $G_1=f_1^{-1}(F_1)=f_1^{-1}(E_{11}\cap E_{12})=f_1^{-1}(E_{11})\cap f_1^{-1}(E_{12})$

(6) $G=f^{-1}(E_1\cup E_2)=f^{-1}(E_1)\cup f^{-1}(E_2)$

(7) $\mu_{1G_1}: G_1 \to L_G$ is defined by $\mu_{1G_1}x = \begin{cases} \mu_{1B_1}x, \text{whenever } \Phi^{-1}\mu_{1F_1}y = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1F_1}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1F_1}y\right)\right], x \in B \\ \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1F_1}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1F_1}y\right), x \notin B \end{cases}$

(8)$\mu_{2G}: G \to L_G$ is defined by $\mu_{2G}x = \begin{cases} \mu_{2B}x \quad, \text{whenever } \Phi^{-1}\mu_{2F}y = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{2F}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{2F}y\right)\right] \end{cases}$

(9)$L_G = L_B$

Let, $\overline{f}^{-1}(\mathcal{E}_1) = \mathcal{D}_1 = \left(D_{11}, D_1, \overline{D}_1\left(\mu_{1D_{11}}, \mu_{2D_1}\right), L_{D_1}\right)$, and $\overline{f}^{-1}(\mathcal{E}_2) = \mathcal{D}_2 = \left(D_{12}, D_2, \overline{D}_2\left(\mu_{1D_{12}}, \mu_{2D_2}\right), L_{D_2}\right)$

where

(10) $D_{11} = f_1^{-1}(E_{11})$

(11)$D_1 = f^{-1}(E_1)$

(12)$\mu_{1D_{11}}: D_{11} \to L_{D_1}$ is defined by $\mu_{1D_{11}}x = \begin{cases} \mu_{1B_1}x, \text{whenever } \Phi^{-1}\mu_{1E_{11}}y = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right)\right], x \in B \\ \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right), x \notin B \end{cases}$

(13) $\mu_{2D_1}: D_1 \to L_{D_1}$ is defined by $\mu_{2D_1}x = \begin{cases} \mu_{2B}x \quad, \text{whenever } \Phi^{-1}\mu_{2E_1}y = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{2E_1}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{2E_1}y\right)\right] \end{cases}$

(14)$L_{D_1} = L_B$

(15) $D_{12} = f_1^{-1}(E_{12})$

(16) $D_2 = f^{-1}(E_2)$

(17) $\mu_{1D_{12}}: D_{12} \to L_{D_2}$ is defined by $\mu_{1D_{12}}x = \begin{cases} \mu_{1B_1}x, \text{whenever } \Phi^{-1}\mu_{1E_{12}}y = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)\right], x \in B \\ \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right), x \notin B \end{cases}$

(18) $\mu_{2D_2}: D_2 \to L_{D_2}$ is defined by $\mu_{2D_2}x = \begin{cases} \mu_{2B}x \quad, \text{whenever } \Phi^{-1}\mu_{2E_2}y = \Phi \\ \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{2E_2}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{2E_2}y\right)\right] \end{cases}$

(19) $L_{D_2} = L_B$

Suppose $\overline{f}^{-1}(\mathcal{E}_1) \cap \overline{f}^{-1}(\mathcal{E}_2) = \mathcal{D}_1 \cap \mathcal{D}_2 = \mathcal{H} = \left(H_1, H, \overline{H}\left(\mu_{1H_1}, \mu_{2H}\right), L_H\right)$, where

(1) $H_1 = D_{11} \cap D_{12} = f_1^{-1}(E_{11}) \cap f_1^{-1}(E_{12})$

(2) $H = D_1 \cup D_2 = f^{-1}(E_1) \cup f^{-1}(E_2)$

(3) $L_H = L_{D_1} \wedge L_{D_2} = L_B$

(4) $\mu_{1H_1}: H_1 \longrightarrow L_H$ is defined by $\mu_{1H_1}x = \left(\mu_{1D_{11}} \wedge \mu_{1D_{12}}\right)x = \mu_{1D_{11}}x \wedge \mu_{1D_{12}}x$

(5) $\mu_{2H}: H \longrightarrow L_H$ is defined by $\mu_{2H}x = \left(\mu_{2D_1} \vee \mu_{2D_2}\right)x = \mu_{2D_1}x \vee \mu_{2D_2}x$,
$x \in D_1 \cup D_2 = B$

Need to show that, $\mathcal{G}$ and $\mathcal{H}$ are full-equal i.e.

(6) $G_1 = H_1, G = H$

(7) $L_G = L_H$

(8) $\mu_{1G_1}x = \mu_{1H_1}x$, $\mu_{2G}x = \mu_{2H}x$

(25) follows from (5), (6), (20) and (21)
(26)follows from (7) and (22)
Sufficient to show that, $\mu_{1G_1}x = \mu_{1H_1}x$, $\mu_{2G}x = \mu_{2H}x$

$\mu_{1H_1}x = \left(\mu_{1D_{11}} \wedge \mu_{1D_{12}}\right)x = \mu_{1D_{11}}x \wedge \mu_{1D_{12}}x$

Case(i) For $x \in B$, $y \in E_{11} \cap E_{12}$

$$\mu_{1H_1}x = \left\{\mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right)\right]\right\} \wedge \left\{\mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)\right]\right\}$$

$$= \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left\{\left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right) \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)\right\}\right]$$

$$\mu_{1G_1}x = \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1F_1}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1F_1}y\right)\right]$$

$$= \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1F_1}y \\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{1E_{11}} \wedge \mu_{1E_{12}}\right)y\right)\right]$$

$$= \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1F_1}y \\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{1E_{11}}y \wedge \mu_{1E_{12}}y\right)\right)\right]$$

$$\bigvee_{\substack{\Phi\alpha=\mu_{1F_1}y \\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{1E_{11}}y \wedge \mu_{1E_{12}}y\right)$$

$$= \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right) \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right) \text{ (}\because\text{Prop 1.7 (a))}$$

$$\Rightarrow \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1F_1}y \\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{1E_{11}}y \wedge \mu_{1E_{12}}y\right)\right)$$

$$= \mu_{1B_1}x \wedge \left[\left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right) \vee \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)\right]$$

$$\Rightarrow \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1F_1}y \\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{1E_{11}}y \wedge \mu_{1E_{12}}y\right)\right)\right]$$

$$= \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left\{\left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right) \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)\right\}\right]$$

$$\Rightarrow \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1F_1}y \\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{1E_{11}} \wedge \mu_{1E_{12}}\right)y\right)\right]$$

$$= \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left\{\left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right) \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)\right\}\right]$$

$\Rightarrow \mu_{1G_1}x = \mu_{1H_1}x$

Case (II): (a) For $x \notin B, y \in E_{11} \cap E_{12}$

$$\bigvee_{\substack{\Phi\alpha=\mu_{1F_1}y \\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{1E_{11}}y \wedge \mu_{1E_{12}}y\right)$$

$$= \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right) \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)$$

$$\Rightarrow \mu_{1B_1}x \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1F_1}y \\ \alpha\in L_B}} \Phi^{-1}\left(\mu_{1E_{11}}y \wedge \mu_{1E_{12}}y\right)\right)$$

$$= \mu_{1B_1}x \wedge \left[\left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{11}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{11}}y\right) \wedge \left(\bigvee_{\substack{\Phi\alpha=\mu_{1E_{12}}y \\ \alpha\in L_B}} \Phi^{-1}\mu_{1E_{12}}y\right)\right]$$

$\Rightarrow \mu_{1H_1}x = \mu_{1G_1}x$

Case (III): For $x \in B, y \in E_1 \cup E_2 = C$

$\mu_{2H}x = \mu_{2D_1}x \vee \mu_{2D_2}x$

$$= \left\{\mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\vee_{\substack{\Phi\alpha=\mu_{2E_1}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{2E_1}y\right)\right]\right\} \vee \left\{\mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\vee_{\substack{\Phi\alpha=\mu_{2E_2}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{2E_2}y\right)\right]\right\}$$

$$= \mu_{2B}x \left[\mu_{1B_1}x \wedge \left(\left(\vee_{\substack{\Phi\alpha=\mu_{2E_1}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{2E_1}y\right) \vee \left(\vee_{\substack{\Phi\alpha=\mu_{2E_2}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{2E_2}y\right)\right)\right]$$

Case (VIII): $x \in B, y \in E_1, y \notin E_2$

$$\mu_{2H}x = \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\vee_{\substack{\Phi\alpha=\mu_{2E_1}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{2E_1}y\right)\right] = \mu_{2F}x$$

Case (IX): $x \in B, y \in E_2, y \notin E_1$

$$\mu_{2H}x = \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge \left(\vee_{\substack{\Phi\alpha=\mu_{2E_2}y\\ \alpha\in L_B}} \Phi^{-1}\mu_{2E_2}y\right)\right] = \mu_{2F}x$$

Hence $\overline{f}^{-1}(\mathcal{E}_1 \cap \mathcal{E}_2)$ and $\overline{f}^{-1}(\mathcal{E}_1) \cap \overline{f}^{-1}(\mathcal{E}_2)$ are full-equal.

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