TRANSIENT STATE SOLUTION OF M/M/C RETRIAL QUEUE AND FEEDBACK ON NON – RETRIAL CUSTOMERS WITH CATASTROPHES

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Abstract: In this paper we study the transient state solution of M/M/C Retrial Queue and Feedback on Non – Retrial Customers under the catastrophic effect. We use generating function technique and we derive the stationary probability distribution. Numerical example is also given to test the feasibility of this model.
Keywords: Transient state, Retrial queue, Feedback queue, Steady state probability, Catastrophes, Stationary probability distribution, Multi server queue. .

Introduction: In the year of 1909, queueing theory originated in telephony with the work of Erlang [3]. After his work many authors to develop different types of queueing models, incorporating different arrival patterns, different service time distributions and various service disciplines. In the year of 1963, Takacs [12] first introduced queues with feedback mechanism which includes the possibility for a customer return to the counter for additional service. In the year of 2008, Santhakumaran and Shanmugasundaram [11] have focused to study a Preparatory Work on Arrival Customers with a Single Server Feedback Queue. In 1983, Kulkarni [7] has proposed the queueing system with retrial customers in which the retrial time is an alternate to the classical queuing models. Fayolle [4] has proposed constant retrial policy to study the M/M/s retrial queue where the customers in the retrial group form a queue and only the customer at the head of the orbit queue can request service after exponentially distributed retrial time. Faramand [5] calls this discipline a retrial queue with FIFO orbit.

In queueing theory, the time independent solutions only derived for a long time. According to the theory and applications of queueing theory time dependent solution is necessary. Parthasarathy [9] and Parthasarathy and Sharafali [8] have discussed single and multiple server poisson queues of transient state solution in easiest manner. Krishna Kumar and Arivudainambi [6] has proposed a transient state solution for the mean queue size of M/M/1 queueing model when catastrophes occurred at the service station. Parthasarathy and Sudesh [10] have introduced transient solution of a multiserver poisson queue with N- policy. Al-Seady, El-sherbiny, El-Sherhawy and Ammar [1] have discussed transient solution of the M/M/C queue with balking and reneging. Darmaraja and Rakesh Kumar [2] have studied transient solution of a Markovian queueing model with heterogeneous servers and catastrophes.In this paper we analyze the time dependent solution of M/M/C feedback queue with catastrophes.

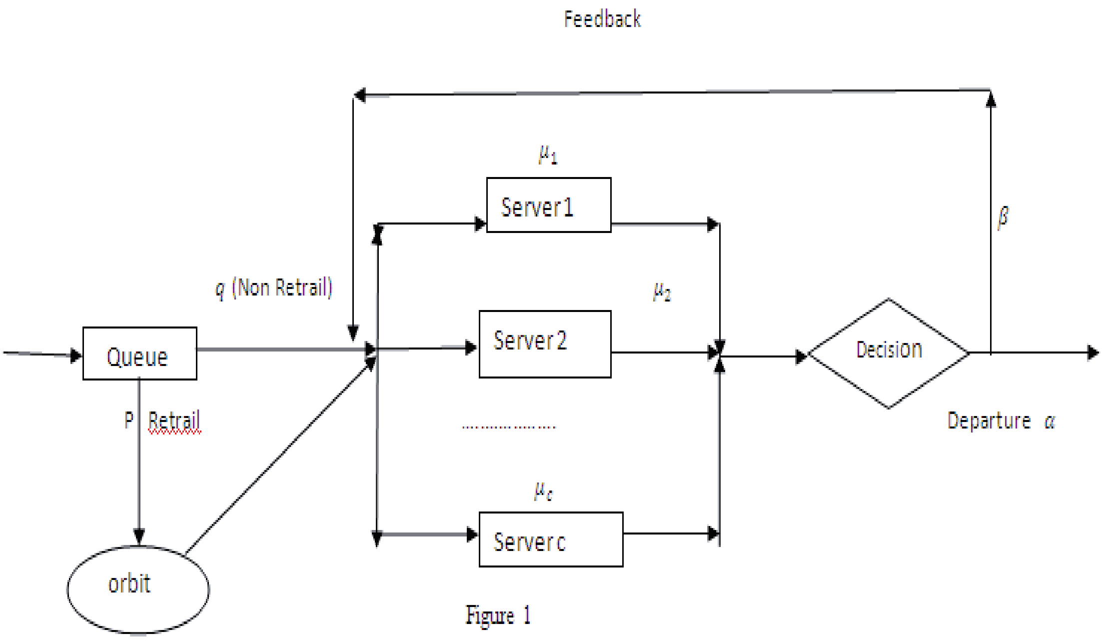


Figure 1

Model Description and Analysis: Fig.1, illustrates the flow of customers through the queueing system. The queueing system is denoted by M/M/C heterogeneous queue with instantaneous Bernoulli feedback and retrial queue. Here the arrival rate of the customer enter into the queue is a poisson process with rate λ_t . If the server is idle, service of an arriving customer starts instantaneously. Service rate follows exponential distributions with rate$\mu_i(t)$. The external arrival of customers enter into the queue with probability q. Otherwise the customer join the retrial queue with probability p. We assume only the customer at the head of the queue is allowed to get service in FIFO basis and the capacity is infinite. If the server is busy upon retrial, the customer joins the orbit again. Such a process is repeated until the customer finds the server is idle and gets the requested and required service at the time of a retrial and feedback. The non retrial customers only feedback but retrial customers does not feedback. After receiving service the retrial customer joins the departure process with probability α and leaves the system forever. After getting service the non retrial customer a decision is made whether or not feedback. If the non retrial customer does feedback, he joins the feedback stream with probabilityβ. If the non retrial customer does not feedback, he joins the departure process with probabilityα. The queue discipline is FIFO and infinite in capacity, the service times are non-negative, independent and identically distributed random variables. Catastrophes occurs from the arrival and service process follows poisson process with rate ν_t. All the available customers are destroyed immediately when the catastrophes occurred in the system, the server gets inactivated. The server is ready for service when a new arrival happens. The motivation for this model comes from Hospitals, Production System, Banks, Restaurant, etc. Let $Q_n(t) = Q[X(t) = n]\,, n = 0,1,\ldots$ denote the transient state probabilities of n customers in the system at a time t. Let $Q(x,t)$ be the probability generating function. Generally we assume that there are no customers in the system at a time $t = 0$.

The system of differential – difference equations for the probability $Q_n(t)$ is

$$Q'_0(t) = -(\lambda_t + \nu_t)\, Q_0(t) + \mu_1(t) Q_1(t) + \nu_t \tag{1}$$

$$Q'_n(t) = -\left(\lambda_t + \nu_t + (p+\alpha)\sum_{i=1}^{n}\mu_i(t) + (q+\beta)\sum_{j=1}^{n}\mu_j(t)\right) Q_n(t) + \lambda_t Q_{n-1}(t)$$
$$+\left[(p+\alpha)\sum_{i=1}^{n+1}\mu_i(t) + (q+\beta)\sum_{j=1}^{n+1}\mu_j(t)\right] Q_{n+1}(t),\ 1 \le n \le c-1 \tag{2}$$

$$Q'_c(t) = -\left(\lambda_t + \nu_t + (p+\alpha)\sum_{i=1}^{c}\mu_i(t) + (q+\beta)\sum_{j=1}^{c}\mu_j(t)\right) Q_c(t) + \lambda_t Q_{c-1}(t)$$
$$+\left[(p+\alpha)\sum_{i=1}^{c}\mu_i(t) + (q+\beta)\sum_{j=1}^{c}\mu_j(t)\right] Q_{c+1}(t), \tag{3}$$

$$Q'_n(t) = -\left(\lambda_t + \nu_t + (p+\alpha)\sum_{i=1}^{c}\mu_i(t) + (q+\beta)\sum_{j=1}^{c}\mu_j(t)\right) Q_n(t) + \lambda_t Q_{n-1}(t)$$
$$+\left[(p+\alpha)\sum_{i=1}^{c}\mu_i(t) + (q+\beta)\sum_{j=1}^{c}\mu_j(t)\right] Q_{n+1}(t),\ n = c+1, c+2, \ldots \tag{4}$$

With $Q_n(0) = \delta_{0n}$, Kronecker delta symbol.

The probability generating function $Q(x,t)$ for the transient state probabilities $Q_n(t)$ is given by

$$Q(x,t) = p_c(t) + \sum_{n=1}^{\infty} Q_{n+c}(t)\, x^n \quad ; Q(x,0) = 1 \tag{5}$$

$$p_c(t) = \sum_{n=0}^{\infty} Q_n(t)$$

$$p_c{}'(t) = Q_0{}'(t) + Q_c{}'(t) + \sum_{n=1}^{c-1} Q_n{}'(t) \tag{6}$$

Substituting equations (1), (2) & (3) we get

$$\frac{dp_c(t)}{dt} = -\lambda_t Q_c(t) + \left[(p+\alpha)\sum_{i=1}^{c}\mu_i(t) + (q+\beta)\sum_{j=1}^{c}\mu_j(t)\right] Q_{c+1}(t) - \nu_t p_c(t) + \nu_t \tag{7}$$

Multiply equation (4) by x^n ,we get,

$$\frac{d\left[\sum_{n=1}^{\infty} Q_{n+c}(t) x^n\right]}{dt} = \left[-\left(\lambda_t + \nu_t + (p+\alpha)\sum_{i=1}^{c}\mu_i(t) + (q+\beta)\sum_{j=1}^{c}\mu_j(t)\right) + \left(\lambda_t x + \frac{(p+\alpha)\sum_{i=1}^{c}\mu_i(t) + (q+\beta)\sum_{j=1}^{c}\mu_j(t)}{x}\right)\right]\sum_{n=1}^{\infty} Q_{n+c}(t)\, x^n + \lambda_t x Q_c(t) - \left[(p+\alpha)\sum_{i=1}^{c}\mu_i(t) + (q+\beta)\sum_{j=1}^{c}\mu_j(t)\right] Q_{n+c}(t) \tag{8}$$

Differentiate equation (5) with respect to 't' and substitute equations (7) & (8) we get,

$$\frac{\partial Q(x,t)}{\partial t}=\left[-\left(\lambda_t+\nu_t+(p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)\right)+\left(\lambda_t x+\frac{(p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)}{x}\right)\right]Q(x,t)-$$
$$\left[\left(\lambda_t x+\frac{(p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)}{x}\right)-\left(\lambda_t+(p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)\right)p_c(t)\right]+\lambda_t(x-1)Q_c(t)+\nu_t \quad (9)$$

Using the integrating factor as $e^{-\left[\left(\lambda_t x+\frac{(p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)}{x}\right)-\left(\lambda_t+\nu_t+(p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)\right)\right]t}$

Solving the above first order differential equation (9) we get,

$$Q(x,t)=e^{-\left[\left(\lambda_t x+\frac{(p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)}{x}\right)-\left(\lambda_t+\nu_t+(p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)\right)\right]t}+$$
$$\int_0^t\{\lambda_t(x-1)Q_c(u)-[(\lambda_t x+\frac{(p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)}{x})-(\lambda_t+(p+\alpha)+(q+\beta)\sum_{j=1}^{c}\mu_j(t))p_c(u)]\}e^{-\left[\left(\lambda_t x+\frac{(p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)}{x}\right)-\left(\lambda_t+\nu_t+(p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)\right)\right](t-u)}du$$
$$+\nu_t\int_0^t e^{-\left[\left(\lambda_t x+\frac{(p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)}{x}\right)-\left(\lambda_t+\nu_t+(p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)\right)\right](t-u)}du \quad (10)$$

If $\tau=(p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)$ Let $\gamma=2\sqrt{\tau\lambda_t}$ and $\delta=\sqrt{\frac{\lambda_t}{\tau}}$, then using the modified Bessel function of first kind $I_n(.)$ and the Bessel function properties, we get $e^{-\left[\left(\lambda_t x+\frac{\tau}{x}\right)\right]t}=\sum_{n=-\infty}^{\infty}(\delta s)^n I_n(\gamma t)$

Substitute equation (10) in equation (9), we get,

$$p_c(t)+\sum_{n=1}^{\infty}Q_{n+c}(t)\,x^n=e^{[-(\lambda_t+\nu_t+\tau)]t}\sum_{n=-\infty}^{\infty}(\delta x)^n I_n(\gamma t)+\lambda_t\int_0^t Q_c(u)\,e^{[-(\lambda_t+\nu_t+\tau)](t-u)}\sum_{n=-\infty}^{\infty}(\delta x)^n(\delta)^{-1}[I_{n-1}(\gamma(t-u))-I_n(\gamma(t-u))]du+\int_0^t p_c(u)\,e^{[-(\lambda_t+\nu_t+\tau)](t-u)}\sum_{n=-\infty}^{\infty}\left[-\lambda_t(\delta)^{-1}\left[I_{n-1}(\gamma(t-u))\right]du+(\lambda_t+\tau)I_n(\gamma(t-u))-\tau I_{n+1}(\gamma(t-u))\right]+\nu_t\int_0^t e^{[-(\lambda_t+\nu_t+\tau)](t-u)}\sum_{n=-\infty}^{\infty}(\delta x)^n I_n(\gamma(t-u))du \quad (11)$$

Equating the co efficient of x^n on both sides, and for $n=1,2,3,\ldots.$

$$Q_{n+c}(t)=e^{-(\lambda_t+\nu_t+\tau)t}\delta^n I_n(\gamma t)+\lambda_t\int_0^t e^{[-(\lambda_t+\nu_t+\tau)](t-u)}\,[I_{n-1}(\gamma(t-u))\delta^{n-1}-I_n(\gamma(t-u)\delta^n)]\,Q_c(u)du -$$
$$\int_0^t e^{[-(\lambda_t+\nu_t+\tau)](t-u)}\left[\lambda_t I_{n-1}(\gamma(t-u))\delta^{n-1}-(\lambda_t+\tau)I_n(\gamma(t-u))\delta^n)+\tau\, I_{n+1}(\gamma(t-u))\delta^{n+1})\right]du+\nu_t\int_0^t e^{[-(\lambda_t+\nu_t+\tau)](t-u)}\delta^n I_n(\gamma(t-u))du \quad (12)$$

Substituting $n=0$ and comparing the constant terms of equation we get,

$$p_c(t)=e^{-(\lambda_t+\nu_t+\tau)t}I_0(\gamma t)+\lambda_t\int_0^t e^{[-(\lambda_t+\nu_t+\tau)](t-u)}\left[I_{-1}(\gamma(t-u))\delta^{-1}-I_0(\gamma(t-u))\right]Q_c(u)du-\int_0^t e^{[-(\lambda_t+\nu_t+\tau)](t-u)}\,p_c(u)\left[\lambda_t I_{-1}(\gamma(t-u))-(\lambda_t+\tau)I_0(\gamma(t-u))\right]du+\nu_t\int_0^t e^{[-(\lambda_t+\nu_t+\tau)](t-u)}\,I_0(\gamma(t-u))du \quad (13)$$

The terms in left hand side of equation (13) have no negative powers of z, therefore left hand side replaced by zero and using $I_n(\gamma(t-u))=I_{-n}(\gamma(t-u))$. We have

$$\int_0^t p_c(u)\,e^{[-(\lambda_t+\nu_t+\tau)](t-u)}\,[\lambda_t I_{n+1}(\gamma(t-u))\delta^{n-1}-(\lambda_t+\tau)I_n(\gamma(t-u))\delta^n)+\tau\, I_{n-1}(\gamma(t-u))\delta^{n+1})]du= e^{[-(\lambda_t+\nu_t+\tau)]t}\delta^n I_n(\gamma t)$$
$$+\lambda_t\int_0^t e^{[-(\lambda_t+\nu_t+\tau)](t-u)}Q_c(u)[I_{n+1}(\gamma(t-u))\delta^{n-1}-I_n(\gamma(t-u))\delta^n]du+\nu_t\int_0^t e^{[-(\lambda_t+\nu_t+\tau)](t-u)}I_n(\gamma(t-u))\delta^n du \quad (14)$$

Substituting equation (14) in equation (12) we get

$$Q_{n+c}(t)=n\delta^n\int_0^t e^{[-(\lambda_t+\nu_t+\tau)](t-u)}\frac{I_n(\gamma(t-u))}{t-u}Q_c(u)du \quad (15)$$

The remaining probabilities $Q_n(t)$,n=0,1,2,…c can be obtained by solving the equations (1)& (2).

Equations (1) & (2) can be written in matrix form as $\frac{dQ(t)}{dt}=AQ(t)+\tau Q_c(t)I_1+\nu_t I_2$ (16)

Where $A = \begin{bmatrix} -(\lambda_t+\nu_t) & \mu_1(t) & \cdots & 0 \\ \lambda_t & -(\lambda_t+\nu_t+\mu_1(t)) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \tau \\ 0 & 0 & & -(\lambda_t+\nu_t+\tau) \end{bmatrix}$ And $Q(t) = \left(Q_0(t) \;\; Q_1(t) \;\; \dots \;\; Q_{c-1}(t)\right)^T$, $I_1 = (0 \;\; 0 \;\; \dots \;\; 1)^T$, $I_2 = (1 \;\; 0 \;\; \dots \;\; 0)^T$ are column vector of order c.

Let $Q^*(s) = \left(Q_0{}^*(s) \;\; Q_1{}^*(s) \;\; \dots \;\; Q_{c-1}{}^*(s)\right)^T$ denotes the Laplace transform of $Q(t)$.

Taking Laplace transform of equation (16) we get,

$$Q^*(s) = (sI - A)^{-1}\left\{Q(0) + \tau Q_c{}^*(s) I_1 + \frac{\nu_t}{s} I_2\right\} \tag{17}$$

With $Q(0) = (1 \;\; 0 \;\; \dots \;\; 0)^T$. To find $Q_c{}^*(s)$, if $e = (1 \;\; 1 \;\; \dots \;\; 1)^T{}_{c\times 1}$, then

$$e^T Q^*(s) + Q_c{}^*(s) = p_c{}^*(s) \tag{18}$$

Let $w = \left[(s + \lambda_t + \nu_t + \tau) - \sqrt{(s + \lambda_t + \nu_t + \tau)^2 - \alpha^2}\right]$

$$s(s + \nu_t) p_c{}^*(s) = (s + \nu_t) + s Q_c{}^*(s) \frac{(w - \gamma\delta)}{2} \tag{19}$$

$$Q_c{}^*(s)\left[1 - \frac{1}{(s+\nu_t)}\frac{(w-\gamma\delta)}{2}\right] = \frac{1}{s} + e^T Q^*(s)$$

Substitute equation (18) we get, $Q_c{}^*(s) = \left(\frac{s+\nu_t}{s}\right) \times \frac{1 - s e^T (sI-A)^{-1}\left(Q(0)+\frac{\nu_t}{s} I_2\right)}{(s+\lambda_t+\nu_t) - \frac{w}{2} + (s+\nu_t) e^T (sI-A)^{-1} I_1 \tau}$ (20)

By Raju and Bhat we get the value of the matrix $(sI - A)^{-1}$.

Let us assume that $(sI - A)^{-1} = (a^*_{uv}(s))_{c\times c}$ For $u = 0, 1, 2, \dots c - 1$

$$a^*_{uv}(s) = \begin{cases} \dfrac{1}{(p+\alpha)\sum_{k=1}^{v+1}\mu_k(t) + (q+\beta)\sum_{k=1}^{v+1}\mu_k(t)} \dfrac{g_{c,v+1}(s) g_{u,0}(s) - g_{u,v+1}(s) g_{c,0}(s)}{g_{c,0}(s)} & v = 0, 1, 2, \dots c - 2 \\ \dfrac{g_{u,0}(s)}{g_{c,0}(s)} & v = c - 1 \end{cases} \tag{21}$$

Where $g_{u,u} = 1, \quad u = 0, 1, 2, \dots c - 1$

$$g_{u+1,u} = \frac{s+\lambda_t+\nu_t+(p+\alpha)\sum_{k=1}^{u}\mu_k(t)+(q+\beta)\sum_{k=1}^{u}\mu_k(t)}{(p+\alpha)\sum_{k=1}^{u+1}\mu_k(t)+(q+\beta)\sum_{k=1}^{u+1}\mu_k(t)} \qquad u = 0, 1, 2, \dots c - 2$$

$$g_{u+1,u-v} = \frac{\left(s+\lambda_t+\nu_t+(p+\alpha)\sum_{k=1}^{u}\mu_k(t)+(q+\beta)\sum_{k=1}^{u}\mu_k(t)\right) g_{u,u-v} - \lambda_t g_{u-1,u-v}}{(p+\alpha)\sum_{k=1}^{u+1}\mu_k(t)+(q+\beta)\sum_{k=1}^{u+1}\mu_k(t)} \qquad v \le u, \quad u = 1, 2, \dots c - 2$$

$$g_{c,v} = \begin{cases} \left(s + \lambda_t + \nu_t + (p + \alpha)\sum_{k=1}^{c-1}\mu_k(t) + (q + \beta)\sum_{k=1}^{u}\mu_k(t)\right) g_{c-1,v} - \lambda_t g_{c-2,v} & , v = 0, 1, 2, \dots c - 2 \\ \left(s + \lambda_t + \nu_t + (p + \alpha)\sum_{k=1}^{c-1}\mu_k(t) + (q + \beta)\sum_{k=1}^{u}\mu_k(t)\right) & , v = c - 1 \end{cases} \tag{22}$$

And $g_{u,v} = 0$ for other u and v. Using the above equations in equation (21), we get

$$Q_c{}^*(s) = \left(\frac{s+\nu_t}{s}\right) \times \frac{1 - (s+\nu_t)\sum_{u=0}^{c-1} a^*_{u,0}(s)}{(s+\lambda_t+\nu_t) - \frac{w}{2} + (s+\nu_t)\sum_{u=0}^{c-1} a^*_{u,c-1}(s)\left((p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)\right)} \tag{23}$$

From equation (18) for $k = 0,1,2, \dots, c - 1$, we get

$$Q_k{}^*(s) = \left(\frac{s+\nu_t}{s}\right) a^*_{k,0}(s) + \left((p + \alpha)\sum_{i=1}^{c}\mu_i(t) + (q + \beta)\sum_{j=1}^{c}\mu_j(t)\right) a^*_{k,c-1}(s) Q_c{}^*(s) \tag{24}$$

We have seen that $a^*_{u,v}(s)$ are all rational algebraic functions in s. The cofactor of the $(u, v)^{th}$ element of $(sI - A)$ is a polynomial of degree $c - 1 - |u - v|$. The characteristic roots of the matrix A are all distinct. Let s_u , $u = 0, 1, \dots c - 1$ be the characteristic roots of the matrix A. Then

$$Q_c{}^*(s) = \frac{\left(1+\frac{\nu_t}{s}\right) B^*(s)}{\frac{w}{2}\left[1 - \frac{2\left((p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)\right)(1-C^*(s))}{\left(s+\lambda_t+\nu_t+(p+\alpha)\sum_{k=1}^{c-1}\mu_k(t)+(q+\beta)\sum_{k=1}^{u}\mu_k(t)\right) - \sqrt{\left(s+\lambda_t+\nu_t+(p+\alpha)\sum_{k=1}^{c-1}\mu_k(t)+(q+\beta)\sum_{k=1}^{u}\mu_k(t)\right)^2 - \gamma^2}}\right]} \tag{25}$$

Where $B^*(s) = \sum_{u=0}^{c-1} \frac{\lim_{s\to s_u}(s - s_u)\left[1 - \sum_{l=0}^{c-1}(s+\nu_t) a^*_{l,0}(s)\right]}{s - s_u}$ (26)

$$C^*(s) = \sum_{u=0}^{c-1} \frac{\lim_{s\to s_u}(s - s_u)\left[\sum_{l=0}^{c-1}(s+\nu_t) a^*_{l,c-1}(s)\right]}{s - s_u} \tag{27}$$

$$Q_c{}^*(s) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(n + 1) \frac{(-1)^m}{\left((p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)\right)} \left(\frac{\gamma}{2\lambda_t}\right)^{n+1} \binom{n}{k} \left(\frac{s+\nu_t}{s}\right) B^*(s)\left(C^*(s)\right)^m$$
$$\frac{\left[\left(s+\lambda_t+\nu_t+(p+\alpha)\sum_{k=1}^{c-1}\mu_k(t)+(q+\beta)\sum_{k=1}^{u}\mu_k(t)\right) - \sqrt{\left(s+\lambda_t+\nu_t+(p+\alpha)\sum_{k=1}^{c-1}\mu_k(t)+(q+\beta)\sum_{k=1}^{u}\mu_k(t)\right)^2 - \gamma^2}\right]^{n+1}}{(n+1)\gamma^{n+1}} \tag{28}$$

By taking inverse Laplace transform, we get

$$Q_c(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(-1)^m}{\left((p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)\right)} \left(\frac{\gamma}{2\lambda_t}\right)^{n+1} \binom{n}{k} \int_0^t B(t - u) \int_0^u C^{c(m)}(u - v)$$
$$e^{-\left(\lambda_t+\nu_t+(p+\alpha)\sum_{k=1}^{c-1}\mu_k(t)+(q+\beta)\sum_{k=1}^{u}\mu_k(t)\right)v} \frac{I_{n+1}(\gamma v)}{v} du dv + \nu_t \int_0^t G(t - u) \int_0^u C^{c(m)}(u - v)$$
$$e^{-\left(\lambda_t+\nu_t+(p+\alpha)\sum_{k=1}^{c-1}\mu_k(t)+(q+\beta)\sum_{k=1}^{u}\mu_k(t)\right)v} \frac{I_{n+1}(\gamma v)}{v} du dv \tag{29}$$

Where $G(t) = \int_0^t B(u)\, du$ and $C^{c(m)}(t)$ is m-fold convolution of C(t) with itself and $C^{c(0)} = \delta(t)$.

$$Q_k(t) = a_{k,0}(t) + \nu_t \int_0^t a_{k,0}(u)\, du + \left((p+\alpha)\sum_{i=1}^{c}\mu_i(t) + (q+\beta)\sum_{j=1}^{c}\mu_j(t)\right)$$
$$\int_0^t a_{k,c-1}(u)\, Q_c(t-u)du, \qquad k = 0, 1, \ldots c-1 \qquad (30)$$

Hence equations (15), (29) & (30) completely determine all the state probabilities.

Particular Case:

Case (i): The server is single server, i.e. when $c = 1$.

From equations (1) & (6) we get $(s + \lambda_t + \nu_t)Q_0{}^*(s) = 1 + \mu_1 Q_1{}^*(s) + \frac{\nu_t}{s}$

$p_1{}^*(s) = Q_0{}^*(s) + Q_1{}^*(s)$

Substituting the above equations in equation (19), we get

$$Q_1{}^*(s) = \left(\frac{s+\nu_t}{s}\right)\frac{1}{\mu_1}\sum_{n=0}^{\infty}\left(\frac{\gamma}{2\lambda_t}\right)^{n+1}(n+1)\left(\frac{\lambda_t}{(s+\lambda_t+\nu_t)}\right)^{n+1}\frac{\left[(s+\lambda_t+\nu_t+\mu_1)-\sqrt{(s+\lambda_t+\nu_t+\mu_1)^2-\gamma^2}\right]^{n+1}}{(n+1)\gamma^{n+1}} \qquad (31)$$

By taking inverse Laplace transform, we get

$$Q_1(t) = \frac{1}{\mu_1}\sum_{n=0}^{\infty}\left(\frac{\gamma}{2\lambda_t}\right)^{n+1}(n+1)\int_0^t \lambda_t{}^{n+1}\frac{u^n e^{-(\lambda_t+\nu_t)u}}{n!}e^{-(\lambda_t+\nu_t+\mu_1)(t-u)}\frac{I_{n+1}(\gamma(t-u))}{(t-u)}du$$
$$+\frac{\nu_t}{\mu_1}\sum_{n=0}^{\infty}\left(\frac{\gamma}{2\lambda_t}\right)^{n+1}(n+1)\int_0^t \lambda_t{}^{n+1}K_1(u)e^{-(\lambda_t+\nu_t+\mu_1)(t-u)}\frac{I_{n+1}(\gamma(t-u))}{(t-u)}du \qquad (32)$$

Where $K_1(t) = \int_0^t u^n \frac{e^{-(\lambda_t+\nu_t)u}}{n!}du$.For $n = 0$ from equation (30), we get

$$Q_0(t) = e^{-(\lambda_t+\nu_t)t} + \mu_1\int_0^t e^{-(\lambda_t+\nu_t)(t-u)}Q_1(u)du + \nu_t\int_0^t e^{-(\lambda_t+\nu_t)u}du \qquad (33)$$

For $n = 1, 2, \ldots$ from equation (15), we get

$$Q_{n+1}(t) = n\delta^n\int_0^t e^{-(\lambda_t+\nu_t+\mu_1)(t-u)}\frac{I_n(\gamma(t-u))}{(t-u)}Q_1(u)du \qquad (34)$$

Case (ii): When $\sum_{i=1}^{c}\mu_i(t) = \sum_{j=1}^{c}\mu_j(t) = \sum_{j=1}^{c}\mu_i$, $p = 0, q = 1$ & $\beta = 0, \alpha = 1$ i.e., there is a customer without feedback and retrial then

$$Q_k(t) = a_{k,0}(t) + \nu_t\int_0^t a_{k,0}(u)\,du + \sum_{i=1}^{c}\mu_i\int_0^t a_{k,c-1}(u)\,Q_c(t-u)du, \; k = 0, 1, \ldots c-1 \qquad (35)$$

This result is exactly coinciding with the paper [2].

Stationary Probability Distribution:

The Stationary Probability Distribution for $\nu_t > 0$ is

$$\pi_c = \lim_{t\to\infty} Q_c(t) = \lim_{s\to 0} s\, Q_c{}^*(s)$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(n+1)\frac{(-1)^m}{\left((p+\alpha)\sum_{i=1}^{c}\mu_i(t)+(q+\beta)\sum_{j=1}^{c}\mu_j(t)\right)}\left(\frac{\gamma}{2\lambda_t}\right)^{n+1}\binom{n}{k}\left(\frac{s+\nu_t}{s}\right)B^*(s)\left(C^*(s)\right)^m$$
$$\frac{\left[\left(\lambda_t+\nu_t+(p+\alpha)\sum_{k=1}^{c-1}\mu_k(t)+(q+\beta)\sum_{k=1}^{u}\mu_k(t)\right)-\sqrt{\left(\lambda_t+\nu_t+(p+\alpha)\sum_{k=1}^{c-1}\mu_k(t)+(q+\beta)\sum_{k=1}^{u}\mu_k(t)\right)^2-\gamma^2}\right]^{n+1}}{(n+1)\gamma^{n+1}}$$

Let c = 1 we get, $\pi_1 = \lim_{t\to\infty} Q_1(t) = \lim_{s\to 0} s\, Q_1{}^*(s)$

$$= \left(\frac{s+\nu_t}{s}\right)\frac{1}{\mu_1}\sum_{n=0}^{\infty}\left(\frac{\gamma}{2\lambda_t}\right)^{n+1}(n+1)\left(\frac{\lambda_t}{(s+\lambda_t+\nu_t)}\right)^{n+1}\frac{\left[(s+\lambda_t+\nu_t+\mu_1)-\sqrt{(s+\lambda_t+\nu_t+\mu_1)^2-\gamma^2}\right]^{n+1}}{(n+1)\gamma^{n+1}}$$
$$= \frac{\nu_t}{\mu_1}\sum_{n=0}^{\infty}\left[\frac{(\lambda_t+\nu_t+\mu_1)-\sqrt{(\lambda_t+\nu_t+\mu_1)^2-\gamma^2}}{2(\lambda_t+\nu_t)}\right]^{n+1}$$
$$= \frac{\nu_t}{\mu_1}\left[\frac{(\lambda_t+\nu_t+\mu_1)-\sqrt{(\lambda_t+\nu_t+\mu_1)^2-\gamma^2}}{2(\lambda_t+\nu_t)}\right]\left[1-\frac{(\lambda_t+\nu_t+\mu_1)-\sqrt{(\lambda_t+\nu_t+\mu_1)^2-\gamma^2}}{2(\lambda_t+\nu_t)}\right]^{-1}$$

$\pi_1 = \frac{\nu_t}{\mu_1}\frac{\rho}{(1-\rho)}$ Where $\rho = \frac{(\lambda_t+\nu_t+\mu_1)-\sqrt{(\lambda_t+\nu_t+\mu_1)^2-\gamma^2}}{2(\lambda_t+\nu_t)}$

Numerical Analysis: In this section some numerical analysis along the steady state probabilities with the related graph based on different arrival rate with fixed service rate $\mu_1 = 4\, and\, 8$ and catastrophic effect $\nu_t = 0.2, 0.4, 0.6$.

Table 1:

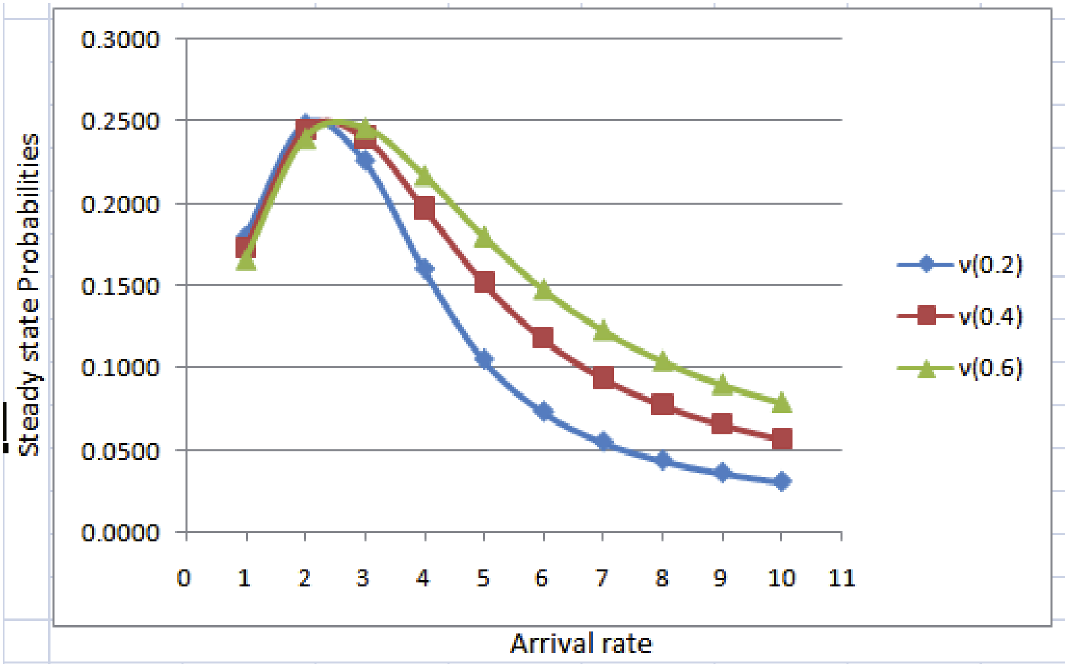
λ	ν(0.2)	ν(0.4)	ν(0.6)
1	0.1796	0.1725	0.1660
2	0.2482	0.2445	0.2400
3	0. 260	0.2400	0.2465
4	0.1600	0.1972	0.2174
5	0.1050	0.1517	0.1802

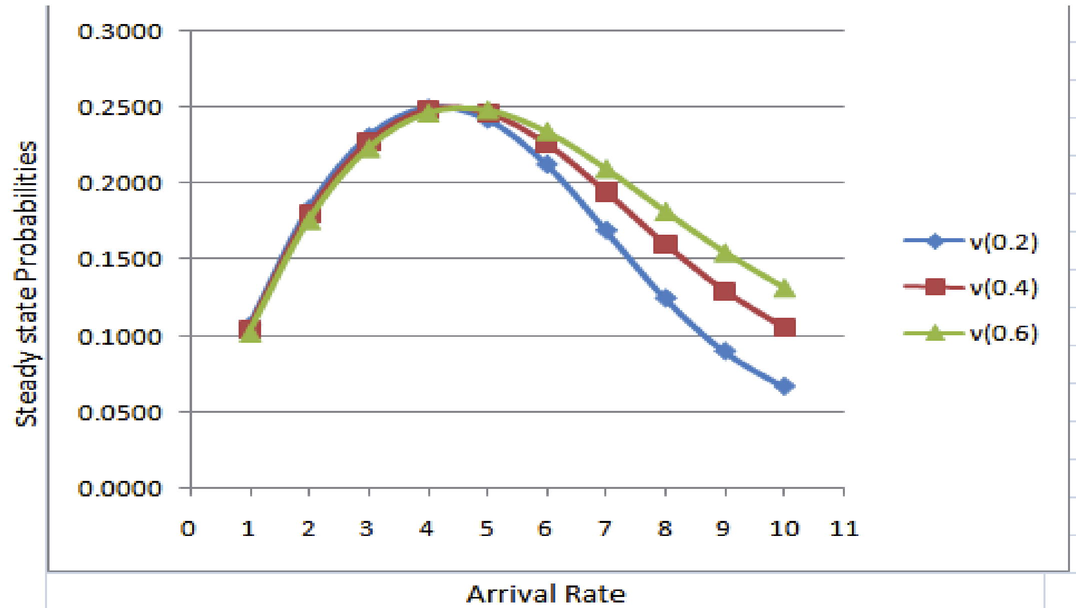
Table 2:

λ	ν(0.2)	ν(0.4)	ν(0.6)
1	0.1068	0.1043	0.1020
2	0.1834	.1796	0.1759
3	0.2306	0.2270	0.2234
4	0.2495	0.2482	0.2465
5	0.2421	0.2463	0.2486
6	0.2126	0.2260	0.2344

6	0.0731	0.1174	0.1480
7	0.0549	0.0938	0.1231
8	0.0436	0.0773	0.1043
9	0.0360	0.0654	0.0900
10	0.0306	0.0566	0.0789

7	0.1692	0.1944	0.2100
8	0.1248	0.1600	0.1818
9	0.0900	0.1293	0.1549
10	0.0668	0.1050	0.1316





Conclusion: The first numerical example shows that if λ increases then steady state probability increases up to $\lambda = 2$ (half of the μ value) and then decreases for different values of ν. The same thing happen in $\mu = 8$, when the starting value occur at $\lambda = 4$. This shows the exactness of the result.

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