A DISCRETE TIME SIS EPIDEMIC MODEL WITH PRIMARY IMMUNODEFICIENCY

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Abstract: In this paper, we construct a discrete time SIS epidemic model where few of the susceptible individuals have low immunity levels. We divide the susceptible population into two groups based on their immunity levels and apply the transmission rate for these two populations. We derive a threshold value known as the basic reproduction number denoted by R_0. We have two equilibria namely the disease free and endemic equilibrium. We analyze the local and global stability of the disease free and endemic equilibrium. Finally, we prove our theoretical results using numerical simulations through MATLAB.

Keywords: Basic reproduction number, Difference equations, Disease-free and endemic equilibria, SIR epidemic model.

1.Introduction: Our immune system is a complicated network of cells, tissues and organs to keep us healthy and fight off diseases and infection. The immune system is composed of two major parts: the innate immune system and adaptive immune system. The macrophages and neutrophils of the innate immune system provide a first line of defense against many common microorganisms and are essential for the control of common bacterial infections. The cells of the innate immune system, however, play a crucial part in the initiation and subsequent direction of adaptive immune responses, as well as participating in the removal of pathogens that have been targeted by an adaptive immune response. Moreover, because there is a delay of 4–7 days before the initial adaptive immune response takes effect, the innate immune response has a critical role in controlling infections during this period.

Immunodeficiency (or immune deficiency) is a state in which the immune system's ability to fight infectious disease is compromised or entirely absent.

Primary immunodeficiencies are disorders in which part of the body's immune system is missing or does not function normally. To be considered a primary immunodeficiency, the cause of immune deficiency must not be secondary in nature (i.e., caused by other disease, drug treatment, or environmental exposure to toxins). Most primary immunodeficiencies are genetic disorders; the majority are diagnosed in children under the age of one, although milder forms may not be recognized until adulthood. About 1 in 500 people in the United States are born with a primary immunodeficiency. The basic tests performed when an immunodeficiency is suspected should include a full blood count (including accurate lymphocyte and granulocyte counts)and i mmunoglobulin levels (the three most important types of antibodies: IgG, IgA and IgM).

People with primary immunodeficiencies are more prone to infections. So in case of an epidemic, people with primary immunodeficiencies are more likely to be infected than other people [4],[5].

An epidemic model with varying immunity period was studied using distributed time delay in continuous time by K.B. Blyuss and Y.N. Kyrychko [5]. In this paper, we have constructed a discrete time epidemic model with primary immunodeficiency and have analyzed the model.

2. Mathematical Model: Before constructing the model, we make the following assumptions:

- $\alpha_1 < \alpha$ .That is, the immunity level of the people in compartment A (α_1) is less than the immunity level of the people in compartment S (α)
- Due to mixing of population, the people of compartment A come in contact with the people in compartment S at a rate q.
- $\beta_1 > \beta$. The transmission rate of people in compartment A is greater than the transmission rate of people in compartment S.

We construct a mathematical model using a system of Difference equations which is given below [1]

$$S(t+1) = bp\theta + (1-\mu)S(t) - \frac{\beta\alpha qS(t)I(t)}{1+cI(t)} + \gamma I(t)$$

$$A(t+1) = b(1-p)\theta + (1-\mu)A(t) - \frac{\beta_1\alpha_1 A(t)I(t)}{1+cI(t)}$$

$$I(t+1) = I(t) + \frac{\beta\alpha qS(t)I(t)}{1+cI(t)} + \frac{\beta_1\alpha_1 A(t)I(t)}{1+cI(t)} - (\mu+\gamma+\delta)I(t)$$

(1)

We have divided the model into four classes: $S(t)$ denotes the Susceptible members of the population without primary immunodeficiencies, $A(t)$ denotes the Susceptible members of the population with primary immunodeficiencies, $I(t)$ denotes the members of the population who are infected by the disease. b is the birth rate of the population. α and α_1 denote the immunity rate of population S and A respectively. θ denote the constant population. μ is the natural death rate. δ is the death due to infection. q is the rate at which population A comes in contact with population S. c is the saturation constant. γ is the rate at which the recovered individuals return to susceptible compartment. β and β_1 are the transmission rates of S and A respectively. p is denotes rate of population without primary immunodeficiency.

Let us take $N(t) = S(t) + A(t) + I(t)$, adding all the equations of the model, we get

$$N(t) = b(p + p_1) - \mu N(t) - \delta I(t) \le b(p + p_1) - \mu N(t)$$

(2)

For our model, we get the equilibrium point $N^* = \frac{b(p + p_1)}{\mu}$, which is globally asymptotically stable as $\lim_{t\to\infty} N(t) = N^*$.The initial conditions are given by,

$$S(0) \ge 0, A(0) \ge 0, I(0) \ge 0$$

Let us assume that all the parameters lie between 0 and 1 should satisfy the following conditions

$$0 < \mu + \gamma + \delta < 2$$

(3)

3. Basic Reproduction Number:

The basic reproduction number, denoted by R_0, is a significant epidemiological quantity, which plays an important role in the dynamics of disease transmission. The basic reproduction number of the model is given by [2],[3]

$$R_0 = \frac{\beta bp\theta q\alpha + \beta_1 b(1-p)\theta\alpha_1}{\mu + \gamma + \delta} \qquad (4)$$

4. Equilibrium Points

- **Disease-free Equilibrium**: Disease free equilibrium is the condition in which the disease dies out in the population . $E^0 = \left(\frac{bp}{\mu}, \frac{bp_1}{\mu}, 0\right)$
- **Endemic equilibrium**: Endemic equilibrium is the condition in which the disease persists in the population. $E^* = (S^*, A^*, I^*)$

Where $S^* = \frac{bp(1+cI^*)}{\beta q\alpha I^* + \mu(1+cI^*)}, A^* = \frac{bp_1(1+cI^*)}{\beta_1\alpha_1 I^* + \mu(1+cI^*)}$

And I^* is a positive root of the following quadratic equation $A_1 I^{*2} + A_2 I^* + A_3 = 0$

$$A_1 = \frac{\beta\beta_1 p + \beta pc\mu + \beta_1 c\mu + c^2\mu^2}{\mu^2}$$

Where $A_2 = \frac{\beta p\mu + \beta_1\mu + 2c\mu^2}{\mu^2} - cR_0 - \frac{\beta\beta_1 pb(A\alpha + A_1\alpha_1)}{\mu^2(\mu + \gamma + \delta)}$

$$A_3 = 1 - R_0$$

5. Local Stability Analysis:

Theorem 5.1: The disease free equilibrium E^0 is locally asymptotically stable if the condition (3) is satisfied. Otherwise unstable.

Proof: Consider the Jacobian matrix of system (1) under E^0 .Consider $|J^0 - \lambda I| = 0$. We get the eigen values $\lambda_1 = 1-\mu, \lambda_2 = 1-\mu, \lambda_3 = 1-(\mu+\gamma+\delta)$ .By the condition in (3), we see that the eigen values $|\lambda_i| < 1$ for i=1,2,3. Therefore the Disease free equilibrium is locally aymptotically stable under condition (3). Otherwise unstable [3].

Theorem 5.2: The Endemic Equilibrium E^* is locally asymptotically stable if the conditions

$$\mu+\frac{\beta q\alpha I^*}{1+cI^*}>1,\ \mu+\delta-\frac{\beta q\alpha S^*}{\left(1+cI^*\right)^2}>1$$

$$(\mu+\delta)+\frac{\beta q\alpha S^*}{\left(1+cI^*\right)^2}>1$$

are satisfied. Otherwise unstable.

Proof: Consider the Jacobian matrix of system (1) under E^*. Consider $\left|J^*-\lambda I\right|=0$ .We get,

$$\phi(\lambda)=\lambda^3+\Omega_1\lambda^2+\Omega_2\lambda+\Omega_3=0 \tag{5}$$

$$\Omega_1=-\left\{1-(\mu+\alpha+\theta)+\left[1-\left(\mu+\frac{\beta q\alpha I^*}{1+cI^*}\right)\right]+\left[1-\left(\mu+\delta-\frac{\beta q\alpha S^*}{\left(1+cI^*\right)^2}\right)\right]\right\}$$

$$\Omega_2=\left[1-(\mu+\alpha+\theta)\right]\left[1-\left(\mu+\frac{\beta q\alpha I^*}{1+cI^*}\right)\right]+\left[1-(\mu+\alpha+\theta)\right]$$

$$\left[1-\left(\mu+\delta-\frac{\beta q\alpha S^*}{\left(1+cI^*\right)^2}\right)\right]\left[1-\left(\mu+\frac{\beta q\alpha I^*}{1+cI^*}\right)\right]\left[1-\left(\mu+\delta-\frac{\beta q\alpha S^*}{\left(1+cI^*\right)^2}\right)\right]$$

$$\Omega_3=-\left[1-(\mu+\alpha+\theta)\right]\left[1-\left(\mu+\frac{\beta q\alpha I^*}{1+cI^*}\right)\right]\left[1-\left(\mu+\delta-\frac{\beta q\alpha S^*}{\left(1+cI^*\right)^2}\right)\right]$$

$$\phi(1)=1+\Omega_1+\Omega_2+\Omega_3>0$$

Then we see that, $\phi(-1)=-1+\Omega_1-\Omega_2+\Omega_3<0$

$$\left|\det J^*\right|<1$$

$$\mu+\frac{\beta q\alpha I^*}{1+cI^*}>1,\ \mu+\delta-\frac{\beta q\alpha S^*}{\left(1+cI^*\right)^2}>1$$

If the conditions

$$(\mu+\delta)+\frac{\beta q\alpha S^*}{\left(1+cI^*\right)^2}>1$$

and the condition (3) are satisfied. It follows from the jury's conditions , that the modulus of all the roots of the above characteristic equation is less than 1, if and only if the conditions $\phi(1)>0,\phi(-1)<0$ and $\left|\det J^*\right|<1$ hold. Hence the proof [5].

Global Stability Analysis:

Let us define the Lyapunov function, $U(t)=g\left(\frac{S(t)}{S}\right)+g\left(\frac{A(t)}{A}\right)+g\left(\frac{I(t)}{I}\right)$ (6)

Where $g(x)=x-1-In(x)\geq g(1)=0$ is defined for $x>0$. We consider the lyapunov functions $U^0(t)$ and $U^*(t)$ to prove the global asymptotic stability of the disease-free equilibrium and endemic equilibrium respectively.

$$U^0(t) = \lim_{I\to 0, S\to S^0, A\to A^0} U(t)$$
$$U^0(t) = g\left(\frac{S(t)}{S^0}\right) + g\left(\frac{A(t)}{A^0}\right) + I(t)$$
$$U^*(t) = \lim_{I\to I^*, S\to S^*, A\to A^*} U(t)$$
$$U^*(t) = g\left(\frac{S(t)}{A^*}\right) + g\left(\frac{A(t)}{A^*}\right) + g\left(\frac{I(t)}{I^*}\right) \quad (7)$$

Theorem 6.1: The disease-free equilibrium E^0 of system (1) is globally asymptotically stable if $R_0 \le 1$.

Proof: Let us consider the Lyapunov function

$$U^0(t) = g\left(\frac{S(t)}{S^0}\right) + g\left(\frac{A(t)}{A^0}\right) + I(t) \quad (8)$$
$$U^0(t) = U_1^0(t) + U_2^0(t) + I(t)$$
$$U_1^0(t+1) - U_1^0(t) = \left[-\frac{\beta p S(t) I(t)}{1 + cI(t)} - \mu\left(S(t) - S^0\right)\right] \frac{\left[S(t+1) - S^0\right]}{S^0 S(t+1)}$$
$$U_2^0(t+1) - U_2^0(t) = \left[-\frac{\beta_1 A(t) I(t)}{1 + cI(t)} - \mu\left(A(t) - A^0\right)\right] \frac{\left[A(t+1) - A^0\right]}{A^0 A(t+1)}$$
$$I(t+1) - I(t) = \frac{\beta\alpha q S(t) I(t)}{1 + cI(t)} + \frac{\beta_1 \alpha_1 A(t) I(t)}{1 + cI(t)} - (\mu + \gamma + \delta) I(t) \quad (9)$$

Substituting the equations (9) in (8) implies, $U^0(t+1) - U^0(t) \le 0$. Then, we have $\lim_{t\to\infty} U^0(t+1) - U^0(t) = 0$, from which we obtain that $\lim_{t\to\infty} I(t+1) = 0, \lim_{t\to\infty} S(t+1) = S^0, \lim_{t\to\infty} A(t+1) = A^0$ for all $t \ge 0$. Hence, the disease free equilibrium E^0 is globally asymptotically stable [2].

Theorem 6.2: If $R_0 > 1$, then the endemic equilibrium E^* of system (1) is globally asymptotically stable.

Proof:

Let us consider the Lyapunov function

$$U^*(t) = g\left(\frac{S(t)}{S^*}\right) + g\left(\frac{A(t)}{A^*}\right) + g\left(\frac{I(t)}{I^*}\right)$$
$$U^*(t) = U_1^*(t) + U_2^*(t) + U_3^*(t) \quad (10)$$

We have,

$$U_1^*(t+1)-U_1^*(t)=\left[bp\alpha\left[1-\frac{S(t)}{S^*}\right]-\beta pS(t)\left[\frac{I(t)}{1+cI(t)}-\frac{I^*}{1+cI^*}\right]\right]\frac{\left[S(t+1)-S^*\right]}{S^*S(t+1)}$$

$$U_2^*(t+1)-U_2^*(t)=\left[bp_1\alpha_1\left[1-\frac{A(t)}{A^*}\right]-\beta_1 A(t)\left[\frac{I(t)}{1+cI(t)}-\frac{I^*}{1+cI^*}\right]\right]\frac{\left[A(t+1)-A^*\right]}{A^*A(t+1)}$$

$$U_3^*(t+1)-U_3^*(t)=\left[\beta pI(t)\left[\frac{S(t)}{1+cI(t)}-\frac{S^*}{1+cI^*}\right]+\beta_1 I(t)\left[\frac{A(t)}{1+cI(t)}-\frac{A^*}{1+cI^*}\right]\right]\frac{\left[I(t+1)-I^*\right]}{I^*I(t+1)}$$

(11)

Substituting the equations (11) in (10) implies, $U^*(t+1)-U^*(t)\le 0$ .Then, we have $\lim_{t\to\infty} U^*(t+1)-U^*(t)=0$, from which we obtain that

$\lim_{t\to\infty} I(t+1)=I^*, \lim_{t\to\infty} S(t+1)=S^*$

$\lim_{t\to\infty} A(t+1)=A^*$

for all $t\ge 0$. Hence, the endemic equilibrium E^* is globally asymptotically stable [2]

Numerical Simulation: Let us consider the following values:

$b=0.6, p=0.8, A=100, \alpha=0.97, \delta=0.147, \beta=0.15, c=0.23, \beta_1=0.5, \gamma=0.7, \mu=0.2, \alpha_1=0.1, q=0.5$

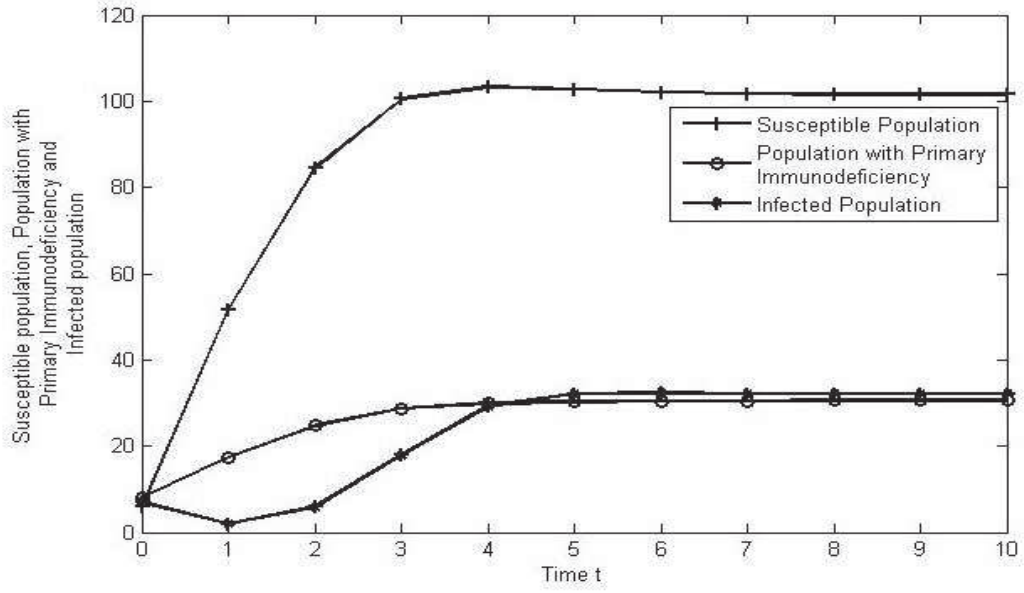

Fig 1: SIR Model with Primary Immunodeficiency *p*=0.8: Now keeping the above set of values, we change the value of $p = 0.91$ which denotes rate of population without primary immunodeficiency. We get the numerical simulation given below:

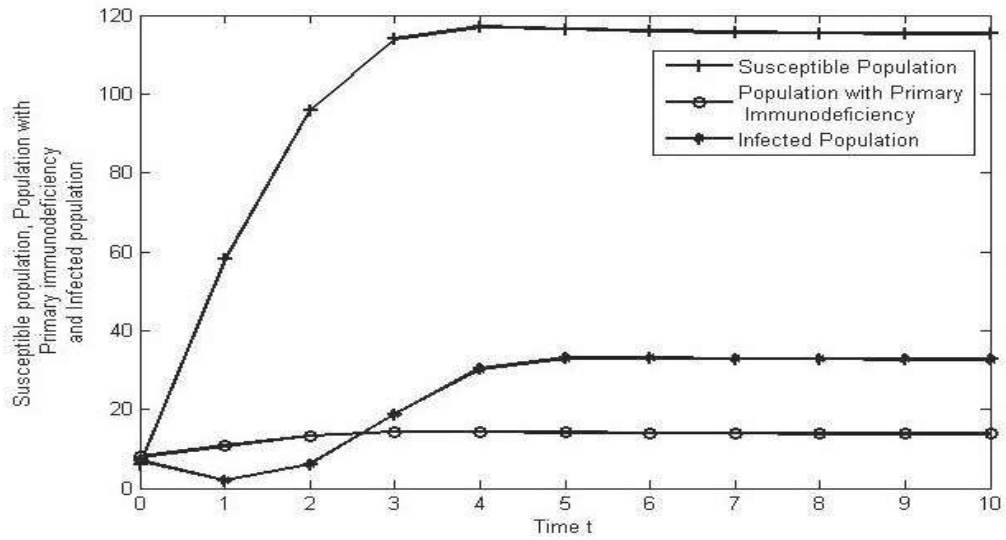

Fig 2: SIR Model with Primary Immunodeficiency *p*=0.91

Conclusion: In this paper, we have proposed a discrete-time SIS epidemic model with primary immunodeficiency and have analyzed its dynamical behavior. We have derived the basic reproduction number R_0 of the model. For this discrete-time model, we have analyzed the local and global asymptotic stability of the disease-free equilibrium and endemic equilibrium for $R_0 \leq 1$ and $R_0 > 1$ respectively. Finally, we have provided numerical simulations for the model using MATLAB.

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