AN ANALYSIS OF EXPONENTIAL STABILITY FOR DIFFERENCE EQUATIONS

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Abstract: In this paper, the stability behavior of exponential stability of dynamical systems of difference equations involving exponential decay has been investigated. For the established dynamical systems of equations, we obtain some conditions and then formulate appropriate inequalities on these functions that guarantee that the zero solution involving decay to attain the exponential stability of the difference equation which involves decay. The obtained result is illustrated through numerical simulation using MATLAB.

Keywords: Difference equation, Exponential Stability, Exponential decay.

1.Introduction: The purpose of this paper is to study the stability behavior of exponential stability for the system of homogeneous difference equations with a non-negative random variable involving exponential decay. The stability of the equation systems is always focused by researchers. A difference equation is a very useful tool in describing and calculating the output of the system described by the formula for a given sample [3]. One of the most important concepts of Digital Signal Processing is to be able to represent the stability of a given system in a proper manner [4].In present days, the structure of Difference equation has plays an important role in real life. It plays a dominant role in designs and applications, for example, frequency-modulated signal processing systems, optimal control models in economics, flying object motions and many evolutionary processes, particularly some biological systems such as biological neural networks and bursting rhythm models in pathology. All these systems are characterized by the fact that at certain moments of time they experience abrupt differences of their states. By overall, difference equation is found in other fields of electronics, information science, and automatic control systems and so on. Equations of Mathematical Physics describe models of some non-steady and non stable physical phenomena like heat flow problem, diffusion and some others. Difference equations are most commonly used to describe the recursive progression of a process or numerical sequence. One of the oldest numerical methods is Euler's method. This method approximates the solution $y(.)$of the first order differential equation

$$y'(t) = f((t)), y(0)\ 0 \le t \le a$$

with a solution $\{y_k\}$ of the following diference equation

$$y_{k+1} = y_k + hf(h_k, y_k), k = 0,1,,2,3 \dots$$

where y_k is an approximation. The accumulated error for this method is of order h whose sample size is . An important part of numerical analysis deals with difference equations that has bounded domains. The related descriptions of finite difference schemes are found in [9].

The paper is organized as follows. In section 2, some important definitions, system formulation and assumptions have been discussed. In section 3, the exponential stability behavior of the homogeneous difference equation involving decay has been arrived. In section 4, numerical simulation is given to illustrate our stability results. Section 5, gives the conclusion of the paper.

2.Preliminaries: To begin with, we introduce some assumptions and recall some basic definitions.

Let B be a Banach space, we will use $|.|$ for the norm in this space and for the induced norm in the space of bounded operators in B, while $||.||$ will be used for the operator norm in some space of function or sequence. Let us consider the sequence $\alpha = \{\alpha_k | k = 1,2, \dots\}$ with $||\alpha|| = sup_{k\ge1}|\alpha_k| < \infty$, and $\alpha = \{\alpha_k | k = 1,2, \dots\} \subset$ B with $||\alpha|| = \sum_{k=1}^{\infty}|\alpha|^q$ $1 \le q <\le \infty$ [3].

In order to clarify the concept of exponential decay, we introduce the weighted y_k spaces as follows. For $y = (y_1, y_2, y_3, \dots, y_s) \in B^s$. For a positive integer , B^s denotes the s-dimensional Euclidean space. Suppose u_k is a sequence on B^s, for $t \ge 0$ and for $1 \le q < \infty$ and set

$$||u_k|| = \left(\sum_{\alpha=1}^{\infty} |u_k(\alpha)e^{t|\alpha|}|^q\right)^{\frac{1}{q}}$$

For $t \ge 0$, let $||u||$ be the supremum of the sequence $|u_k(\alpha)e^{t|\alpha|}|$ and $\alpha \in B^s$.

2.1.System Formulation: Let $\{y_k | k = 1,2,3 \dots\}$ and $\{\gamma_k | k = 1,2,3, \dots\}$ be sequences in B and $A_k: B \to B$ be a bounded operator for $k = 0,1,2,3 \dots$. Now consider a class of difference equations in R^k shown as follows

$$y_{k+1} = A_k y_k\ , k \ge 0 \qquad (1)$$

where $y^* \in R^k$ is the unique equilibrium point and $T: R^k \to R^k$is a continuous differentiable nonlinear operator in R^k , and

$$\lim_{||y||\to\infty} \sup||T'(y)|| = \lambda < 1$$

holds for a positive constant λ ,$0 < \lambda < 1$. The related non homogenous difference equation

$$y_{k+1} = A_k y_k + \gamma_{k+1}, \quad k \ge 0 \qquad (2)$$

and the homogeneous difference equation with an arbitrary initial point n_0 will be [10]

$$y_{k+1} = A_k y_k\ , k \ge n_0\ , \ y_{n_0} = I \qquad (3)$$

Where I is the identity operator which is called as fundamental function of any of the equation (1) and (2).

2.2.Definition: The zero solution of system (1) is said to be:

Stable (S) if given $\varepsilon > 0$ and $n_0 \geq 0$ there exists $\delta = \delta(\varepsilon, n_0)$ such that$||y_0|| < \delta$ implies $\left||y(n, n_0, y_0)\right|| < \epsilon$ for all $n \geq n_0$,

Uniformly stable (US) if δ may be chosen independent of n_0, ***unstable*** if it is not stable.

Attracting (A) if there exists $\mu = \mu(n_0)$ such that $||y_0|| < \mu$ implies

$\lim_{n\to\infty} y(n, n_0, y_0) = 0$,

uniformly attracting (UA) if the choice of μ is independent of n_0. The condition for ***uniform attractivity*** may be paraphrased by saying that there exists $\mu > 0$ such that for every ε and n_0 there exists $N = N(\varepsilon)$ independent of n_0 such that

$$\left||y(n, n_0, y_0)\right|| < \epsilon$$

for all $n \geq n_0 + N$ whenever $||y_0|| < \mu$.

Asymptotically stable **(AS)** if it is stable and attracting, and ***uniformly asymptotically stable*** (UAS) if it is uniformly stable and uniformly attracting.

Exponentially stable **(ES)** if there exist $\delta > 0, M > 0$, and $\eta \in (0,1)$ such that

$$\left||y(n, n_0, y_0)\right|| < M||y_0||\eta^{n-n_0}$$

whenever $||y_0 - y^*|| < \delta$. If $\delta = \infty$, the corresponding stability property is said to be ***global***. And is ***uniformly exponentially*** stable if M is independent of n_0

A solution $y(n, n_0, y_0)$ is ***bounded*** if for some positive constant M, $||y(n, n_0, y_0)|| \leq M$ for all $n \geq n_0$, where M may depend on each solution.

3. Main Results: Our main problem is to find the behavior of the exponential stability behavior, which we can deduce from a theorem. And also the impulse ratio is considered between the systems of difference equations. Let us recall some relevant result o the above kind [3].

Lemma 3.1: Let $\psi_{k,n}$ be a fundamental function of (1), then the solution of (2) can be presented as

$$y_k = \psi_{k,0} y_0 + \sum_{s=0}^{k-1} \psi_{k,s+1} \gamma_{s+1} \qquad (4)$$

Lemma 3.2: Suppose $\sup_k |A_k| < \infty$

(1). Let $1 \leq q \leq \infty$ .Then (1) is exponentially stable if and only if for any $\{\gamma_k\} \in I^q$, the solution of (2), with $y_0 = 0$, belongs to the same space $\{y_k\} \in I^q$.

(2). If $1 \leq q \leq \infty$ and the solution of (2), with $y_0 = 0$, is bounded $\{y_k\} \in I^\infty$ for any $\{\gamma_k\} \in I^q$ then (1) is exponentially stable.

(3). If the solution of (2), with $y_0 = 0$, is bounded for any $\{\gamma_k\} \in I^1$ then (1) is uniformly stable.

Theorem 3.3: If (1) is a fundamental function, then

(1). If $\{\gamma_k\} \in I^\infty$, then the solution is bounded.

(2). If $\lim_{k\to\infty} \gamma_k = \infty$, then the solution of $\{y_k\}$ of (2) satisfies $\lim_{k\to\infty} y_k = 0$

(3). If there exist $M_0 > 0, \lambda > 0$ such that $|\gamma_k| \leq M_0 e^{-\lambda k}$, then there exist $M_1 > 0$, $\eta_1 > 0$ such that the solution $\{y_K\}$ of (2) satisfies $|y_k| \leq M_1 e^{-\eta_1 k}$, which leads the exponential stability.

Proof: Since $\psi_{k,n}$ is the fundamental solution of the problem (1), $\psi_{k,n} y_n$ is a solution of (3) for any $y_n \in B$. Exponential stability of the equation (1) implies that $|\psi_{k,l} y_k| \leq M e^{-\eta(n-k)} |y_k|$ for any $y_k \in B$, Thus the fundamental function has the exponential estimation

$$|\psi_{k,n}| \leq M e^{-\eta(k-n)} \quad n = 0,1,2, \ldots, k \quad (5)$$

(1). The solution $\{y_k\}$ of (2) satisfies (4), therefore

$$|y_k| \leq |\psi_{k,0}|.|y_0| + \sum_{s=0}^{k-1} |\psi_{k,s+1}|.|\gamma_{s+1}| \quad (6)$$

Since $\{\gamma_k\} \in I^\infty$and estimation (6) holds, then we get

$$|y_k| \leq M e^{-\eta k} |y_0| + \sum_{s=0}^{k-1} M e^{-\eta(k-n-1)} |\gamma_{s+1}|$$

$$< M\left(|y_0| + ||\{\gamma_k\}||.\sum_{s=0}^{\infty} e^{-\eta s}\right)$$

$$= M\left(|y_0| + \frac{||\{\gamma_k\}||}{1 - e^{-\eta}}\right)$$

This is true for any positive integer k. Thus any solution $\{y_k\}$ of (2) is bounded.

(2). Let us consider any $y_0 \in B$ and any $\epsilon > 0$. By the assumption of (1), the sequence $\{\gamma_k\}$ converges to zero solution, which implies that $\{\gamma_k\} \in I^\infty$ and there exists $k_1 > 0$ such that

$$|\gamma_k| < \epsilon \frac{1 - e^{-\eta}}{3M} \quad \text{for any } k \geq k_1$$

Since $e^{-\eta(k-n_1-1)} \to 0$ as $k \to \infty$, there exists a positive integer k_2 such that

$$e^{-\eta(k-k_1-1)} < \epsilon \frac{1 - e^{-\eta}}{3M||\{\gamma_k\}||} \quad \text{for any } k \geq k_2$$

Finally, for a given y_0 we can always find positive integer k_3 such that

$$e^{-\eta k} < \frac{\epsilon}{3M|y_0|} \quad \text{for any } k \geq k_3$$

Let us assume that $k > \max\{k_1, k_2, k_3\}$. Since the solution $\{y_k\}$ of (2) satisfies (4) , we get

$$|y_k| \leq |\psi_{k,0}| . |y_0| + \sum_{s=0}^{k-1} |\psi_{k,s+1}|.|\gamma_{s+1}|$$

$$= |\psi_{k,0}|.|y_0 + \sum_{s=0}^{k_1} |\psi_{k,s+1}|.|\gamma_{s+1}|$$

$$+ \sum_{s=k_1+1}^{k-1} |\psi_{k,s+1}|.|\gamma_{s+1}|$$

Applying (6) in the above equation, we get

$$|y_k| \leq M e^{-\eta k} |y_0|$$

$$+ \sum_{s=0}^{k_1} M e^{-\eta(k-s-1)} |\gamma_{s+1}|$$

$$+ \sum_{s=k_1+1}^{k-1} M e^{-\eta(k-s-1)} |\gamma_{s+1}| \quad (7)$$

Let us consider each component of the sum (7). Since $k \geq k_3$, we obtain

$$Me^{-\eta k}\,|y_0| < M\frac{\epsilon}{3M|y_0|}|y_0| = \frac{\epsilon}{3}$$

If $k \geq k_2$ implies,

$$\sum_{s=0}^{k_1} Me^{-\eta(n-s-1)}\,|\gamma_{s+1}|$$

$$= Me^{\eta}||\{\gamma_k\}||\sum_{s=0}^{k_1} e^{-\eta(k-s)}$$

$$= Me^{\eta}\,||\{\gamma_k\}||\sum_{s=k-k_1}^{k} e^{-\eta s}$$

$$< Me^{\eta}\,||\{\gamma_k\}||\sum_{s=k-k_1}^{\infty} e^{-\eta s}$$

$$< Me^{\eta}\,||\{\gamma_k\}||e^{-\eta(k-k_1)}\frac{1}{1-e^{-\eta}}$$

$$< M\,||\{\gamma_k\}||e^{-\eta(k-k_1-1)}\frac{1}{1-e^{-\eta}}$$

$$< M\,||\{\gamma_k\}||\epsilon\frac{1-e^{-\eta}}{3M||\{\gamma_k\}||}\frac{1}{1-e^{-\eta}}$$

$$= \frac{\epsilon}{3}$$

Finally $k \geq K_1$ provides,

$$\sum_{s=k_1+1}^{k-1} Me^{-\eta(k-s)}\,|\gamma_{s+1}|$$

$$< \sum_{s=k_1+1}^{k-1} Me^{-\eta(k-s)}\,\epsilon\frac{1-e^{-\eta}}{3M}$$

$$= \frac{\epsilon}{3}(1-e^{-\eta})\sum_{s=k_1+1}^{k-1} e^{-\eta(k-s-1)}$$

$$= \frac{\epsilon}{3}(1-e^{-\eta})\sum_{s=k_1+1}^{k-1} e^{-\eta(k-s-1)}$$

$$= \frac{\epsilon}{3}(1-e^{-\eta})\sum_{s=0}^{k-k_1} e^{-\eta s}$$

$$< \frac{\epsilon}{3}(1-e^{-\eta})\sum_{s=0}^{k_1} e^{-\eta s}$$

$$< \frac{\epsilon}{3}(1-e^{-\eta})\frac{1}{1-e^{-\eta}} = \frac{\epsilon}{3}$$

From (7), we get,

$$|y_k| \leq Me^{-\eta k}\,|y_0|$$

$$+\sum_{s=0}^{k_1} Me^{-\eta(k-s-1)}\,|\gamma_{s+1}|$$

$$+\sum_{s=k_1+1}^{k-1} Me^{-\eta(k-s-1)}\,|\gamma_{s+1}|$$

$$< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon$$

For any $k > \max\{k_1, k_2, k_3\}$ and $\lim_{k\to\infty} y_k = 0$

(3). The solution $\{y_k\}$ of the equation (1) satisfies (5) therefore

$$|y_k| \leq |\psi_{k,0}|\,.\,|y_0| + \sum_{s=0}^{k-1}|\psi_{k,s+1}|.\,|\gamma_{s+1}|$$

The assumption of Part 3 of this Theorem and (6) together impose,

$$|y_k| \leq Me^{-\eta k}\,|y_0|$$

$$+\sum_{s=0}^{k-1} Me^{-\eta(k-s-1)}\;M_0e^{-\lambda(s+1)}$$

$$|y_k| \leq Me^{-\eta k}\,|y_0|$$

$$+MM_0e^{\eta-\lambda}e^{-\eta k}\sum_{s=0}^{k-1} e^{s(\eta-\lambda)}$$

Without loss of generality we can assume that $\eta \neq \lambda$. Therefore,

$$\sum_{s=0}^{k-1} e^{s(\eta-\lambda)} = \frac{e^{k(\eta-\lambda)}-1}{e^{(\eta-\lambda)}-1}$$

So,

$$Me^{-\eta k}\,|y_0| + MM_0e^{\eta-\lambda}e^{-\eta k}\sum_{s=0}^{k-1} e^{s(\eta-\lambda)}$$

$$= Me^{-\eta k}\,|y_0|$$

$$+ MM_0e^{\eta-\lambda}e^{-\eta k}\frac{e^{k(\eta-\lambda)}-1}{e^{(\eta-\lambda)}-1}$$

$$\leq Me^{-\eta k}\,|y_0| + MM_0e^{\eta-\lambda}e^{-\eta k}\frac{e^{k(\eta-\lambda)}}{e^{(\eta-\lambda)}-1}$$

$$\leq Me^{-\eta k}\,|y_0| + MM_0\frac{e^{\eta-\lambda}}{e^{(\eta-\lambda)}-1}e^{-\lambda k}$$

So, for $\eta_1 = \min\{\eta, \lambda\}$ and $N_3 = M|y_0| + MM_0\frac{e^{\eta-\lambda}}{e^{(\eta-\lambda)}-1}$.

This implies,

$$|y_k| \leq Me^{-\eta k}|y_0|$$

$$+MM_0\frac{e^{\eta-\lambda}}{e^{(\eta-\lambda)}-1}e^{-\lambda k} \qquad (8)$$

for any $k = 0,1,2,3, \ldots$ and hence the solution of (1) is exponential stable.

This completes the proof of the Theorem.

4. Numerical Simulation: The numerical simulation is obtained by collecting the coefficient values of the fundamental function of a system homogeneous difference equation [12] and we compare it with decay and without decay through MATLAB.

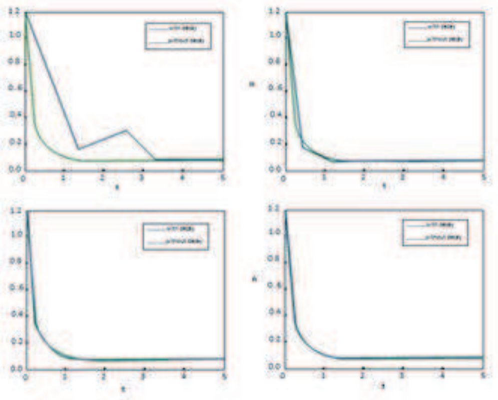

Figure 1: exponential stability observation of the homogeneous equation involving decay

When the coefficients of the fundamental function of homogeneous difference equation lie on the same level then the exponential stability of the equation is evolved. The simulation result shows that the decay rate is reduced to strong to obtain our desired stability result.

5. Conclusion: In this paper, a class of difference equations with decay has been considered. And our objective of attaining the behavior of exponential stability has been obtained by establishing some new necessary conditions, to the system of homogeneous difference equations with distributed decays. And the stability result will be extended to attain uniformly globally exponentially stable in our future work.

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