THE MANIFESTATION OF THE CONNECTEDNESS OF A REAL INTERVAL VIA PATH CONNECTEDNESS

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Abstract: The topological property, Connectedness was studied in this paper. The fact that any interval $I \in \mathbb{R}$ is connected is not entirely obvious, however, in contrast the path connectedness of interval is clear: since any $x, y \in I$,

$$\gamma: [0,1] \to \mathbb{R}, \gamma(t) = (1-t)x + ty$$

takes values in I (since, I is an interval) and connect x and y.
Also, unlike connectedness, path connectedness can be checked more directly, therefore, this idea was used, in this paper to prove some theorems on connectedness (that is to show that path connectedness implies connectedness). Some remarks deduced from the proofs were given as corollaries.

Keywords: (Topological property, Connectedness, Path Connectedness, Real interval)

Introduction : In topology and related areas of mathematics a topological property or topological invariant is a property of a topological space which is invariant under homeomorphisms. That is, a property of spaces is a topological property if whenever a space X possesses that property every space homeomorphic to X possesses that property. Informally, a topological property is a property of the space that can be expressed using open sets.
A common problem in topology is to decide whether two topological spaces are homeomorphic or not. To prove that two spaces are not homeomorphic, it is sufficient to find a topological property which is not shared by them. (Wikipedia)
In this research we take a closer look at properties which depend on the underlying collection of open sets. To do this we introduce the concept of a topological space.
A topological space is a pair (X, τ) consisting of a set X and a collection τ of subsets of X such that;
(i) $\emptyset$ and X are in τ
(ii) if $\{U_\propto\}$ g is a collection of sets in τ then the union $\cup_\propto U_\propto$ is in τ
(iii) if $(U_1, U_2, \ldots, U_n)$ is a finite collection of sets in τ then the intersection $\cap_{j=1}^{n} U_j$is in τ
The collection τ is called a topology on X. The elements of τ are called open sets. While, (X, τ) is called topological space.
A topological space (X,τ) is connected if X cannot be written as the union of two disjoint non-empty open sets $U, V \subset X$.
A topological space (X, τ) is path connected if for any $x, y \in X$, there exists a path γ connecting x and y, i.e. a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$.
Given (X, τ), we say that a subset $A \subset X$ is connected (or path connected) if A, together with the induced topology, is connected (path connected). [6]
Let X and Y denote arbitrary given sets. By a function $F: X \to Y$, we mean a rule which assigns to each element of x of X a unique element y of Y. If $x \in X$, the corresponding element $y \in Y$ is called the f − image of x and is denoted by f(x), i.e., b = f(x).
Let f be a function of X into Y. Then $f(X) \subset Y$, if f (X) = Y and f is a function X onto Y, or $f: X \to Y$ is an onto (surjective)function. the function $f: X \to Y$ is one − one (injective) if for any any two element $x_1$and x_2 of X, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. A function which is both injective and surjective is called bijective [4]
A homeomorphism, also called a continuous transformation, is an equivalence relation and one-to-one correspondence between points in two geometric figures or topological spaces that is continuous in both directions. A homeomorphism which also preserves distances is called an isometry. Affine transformations are another type of common geometric homeomorphism [5]
A research on various topological properties of a k-ary n-cube (Q/sub n//sup k/) using Lee distance and how to find all disjoint paths between any two nodes, if a sequence of radix k numbers is given was conducted by [1]. They also, presented a function mapping the sequence to a Gray code sequence, which was used to generate a Hamiltonian cycle.
Topological formalisms adaptable to a wide range of applications in spatial information systems (SIS) are a highly desirable research goal. In this research, the researcher discusses a candidate formalism, the region-based `logic of connection' of Randell, Cohn and Cui. This is based on the single primitive `C' (where `C(x; y)' indicates that spatial regions x and y are `connected'). The originators of this approach (`RCC theory') defined a set of eight `base relations', exactly one of which holds between any pair of regions. It is shown how finer distinctions between topological relations than those provided by these

base relations, and taxonomies of topological properties of regions, can be defined using RCC theory. [3]

Based on the observation that the category of concept spaces with the positive information topology is equivalent to the category of countable based T_0 topological spaces, we investigate further connections between the learning in the limit model of inductive inference and topology. In particular, we show that the "texts" or "positive presentations" of concepts in inductive inference can be viewed as special cases of the "admissible representations" of computable analysis. We also show that several structural properties of concept spaces have well known topological equivalents. In addition to topological methods, we use algebraic closure operators to analyze the structure of concept spaces, and we show the connection between these two approaches. [2]

Methodology: The following lemma, proposition and theorem together with their proofs and examples that would clarify the context were used:

Lemma 1.1: The unit interval [0, 1] is connected.

Proposition 1.2: (i) If f: X → Y is a continuous map and X is connected, then f(X) is connected.

(ii) Given (X,τ), if for any two points x, y ∈ X, there exists ω⊂ X connected such that x, y ∈ ω, then X is connected.

Theorem 1.3: Any path connected space X is connected.

Lemma 1.1 (Proof): We assume the contrary: ∃ disjoint non-empty U, V, opens in [0, 1] such that U∪V = [0, 1]. Since U = [0, 1] –V, U must be closed in [0, 1]. Hence, as a limit of points in U, ℝ: = supU must belong to U. We claim that ℝ = 1. If not, we find an interval (ℝ $-\varepsilon$, ℝ $+\varepsilon$) ⊂ U and then ℝ + $\frac{1}{2}\varepsilon$ is an element in U strictly greater than its supremum- which is impossible. In conclusion, 1 ∈ U, but exactly the same argument shows that 1 ∈ V, and this contradicts the fact that U ∩ V = ∅.

Proposition 1.2 (proof):

For (i), replacing Y by f(X), we may assume that f is surjective, and we want to prove that Y is connected. If it is not, we find U, V ⊂ Y disjoint nonempty opens whose union is Y. But then f–1 (U), f–1 (V) ⊂ X are disjoint (since U and V are), nonempty (since U and V are and f is surjective) opens (because f is continuous) whose union is X- and this contradicts the connectedness of X. For (ii) we reason again by contradiction, and we assume that X is not connected, i.e. X = U ∪ V for some disjoint nonempty opens U and V. Since they are non-empty, we find x ∈ U, y ∈ V. By hypothesis, we find ω connected such that x, y ∈ ω. But then U′ = U ∩ ω, V ′ = V ∩ ω are disjoint non-empty opens in ω whose union is ω - and this contradicts the connectedness of ω

Theorem 1.3 (Proof):

Using (ii) of the proposition 1.2 and letting x, y ∈ X. Since, there exists;

γ: [0, 1] → X joining x and y. But then ω = γ ([0, 1]) is connected by using (i) of the proposition and the fact that [0, 1] is connected; also, x, y ∈ ω

Unlike connectedness, path connectedness can be checked more directly (see the examples below).

Examples

(1) X = {0, 1} with the discrete topology is not connected. Indeed, U = {0}, V = {1} are disjoint non-empty opens (in X) whose union is X.

(2) Similarly, X = [0, 1) ∪ [2, 3] is not connected (take U = [0, 1), V = [2, 3]). More generally, if X ⊂ ℝ is connected, then X must be an interval. Indeed, if not, we find r, s ∈ X and

t ∈ (r, s) such that t /∈ X. But then U = (−∞, t) ∩ X, V = (t, ∞) ∩ X are opens in X, nonempty (as r ∈ U, s ∈ V), disjoint, with U ∪ V = X (as t /∈ X).

(3) However, although true, the fact that any interval I ⊂ ℝ is connected is not entirely obvious. In contrast, the path connectedness of intervals is clear: for any x, y ∈ I, γ: [0, 1] → ℝ,

γ(t) = (1 − t)x + ty takes values in I (since I is an interval) and connects x and y.

(4) Similarly, any convex subset X ⊂ $\mathbb{R}^n$ is path connected (recall that X being convex means that for any x, y ∈ X, the whole segment [x, y] is contained in X).

Analysis: From the above, we have seen that a connected subset of ℝ must be an interval, and that any interval is path connected, therefore, the theorem implies:

Corollary 1.4: The only connected subsets of ℝ are the intervals.

Combining this corollary with part (i) of Proposition 1.2 we deduce the following:

Corollary 1.5: If X is connected and f: X → ℝ is continuous, then f(X) is an interval.

Corollary 1.6: If X is connected, then any quotient of X is connected.

There are few more consequences that one can derive by combining connectedness properties with the "removing one point trick".

(1) ℝ cannot be homeomorphic to S^1. Indeed, if we remove a point from ℝ the result is disconnected, while if we remove a point from S^1, the result stays connected.

(2) ℝ cannot be homeomorphic to $\mathbb{R}^2$. The argument is similar to the previous one (recall that $\mathbb{R}^2$ − {0} is path connected, hence connected).

(3) The more general statement that $\mathbb{R}^n$ and $\mathbb{R}^m$ cannot be homeomorphic if n ≠m is much more difficult to prove. One possible proof is a generalization of the argument given above (when n = 1, m = 2) - but that is based on "higher versions of connectedness", a notion which is at the core of algebraic topology.

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