HARMONIC POLYNOMIAL AND CHROMATIC HARMONIC POLYNOMIAL AND INDICES FOR LINEAR JACO GRAPH

R. JAYASREE, A. KULANDAI THERESE, U. MARY, JOHAN KOK

Abstract: In this paper the notion of harmonic polynomial and chromatic harmonic polynomial with the corresponding indices are discussed. The minimum and maximum harmonic polynomials, chromatic harmonic polynomials and the respective indices are obtained for linear Jaco graphs $J_n(x), 2 \le n \le 12$. The best fit quadratic progression approximation is also obtained for the above indices.

Keywords: Chromatic harmonic polynomial, Chromatic harmonic index, Harmonic polynomial, Harmonic index and Linear Jaco graph.

1 Introduction: For general notation and concepts in graphs and digraphs see [1]. Unless mentioned otherwise all graphs are simple connected graphs. We will write that a graph G has order $v(V) = n$ and size $\varepsilon(G) = p$ with minimum and maximum degree $\delta(G)$ and $\Delta(G)$, respectively. The quantitative structure-property relationship (QSPR) inherent to chemical structures enjoys much research attention. In earlier years Randic proposed a new structural descriptor called the Randic index. This index was later varied by Fajtlowisc to what is known as the harmonic index. This variation is perhaps the most successful in describing or predicting QSPR and quantitative structure-activity relationships (QSAR). The harmonic index is defined as $H(G) = \sum_{uv \in E(G)} \frac{1}{d_G(u).d_G(v)}$. Zhong [7] determined both minimum and maximum values of the Harmonic index for certain families of graphs. Interesting research results in respect of graphs under pertubations was published by Li and Shiu. Very recently Iranmaneshet. al [2] introduced the concept of the harmonic polynomial of a graph G. The harmonic polynomial is defined as:

$$H(G,x) = \sum_{uv \in E(G)} 2x^{d_G(u)+d_G(v)-1},$$

$$where \int_0^1 H(G,x)dx = H(G)$$

It is observed that most structural indices of kind are defined in terms of the vertex degree in G. The variation we will also consider is that of the colour of a vertex when applying what is known to be a minimum parameter chromatic colouring to G. [3]. Since at least one edge is required we define the default values $H(K_1,x) = 0,$

and $\int_0^1 H(K_1,x)dx = H(G) = 0$

2 Finite Linear Jaco Graphs: The family of trivial finite linear Jaco graphs was introduced by Kok et al [4]. These directed graphs are derived from the infinite linear Jaco graph called the x-root diagraph.

Definition 2.1: The infinite *l*-Jaco graph or root *l*-Jaco graph, $J_\infty(x), x \in N$ is defined by, $V(J_\infty(x)) = \{v_i : i \in N\}$ $A(J_\infty(x)) \subseteq \{(v_i, v_j) : i, j \in N, i < j\}$ and $(v_i, v_j) \in A(J_\infty(x))$ if and only if $2i - d^-(v_i) \ge j.$

Definition 2.2: The family of finite *l*-Jaco graphs is defined by $\{J_n(x) \subseteq J_\infty(x) : n, x \in N\}$. A member of the family is denoted by, $J_n(x).$

Definition 2.3: The set of vertices attaining degree $\Delta(J_n(x))$ is called the set of Jaconian vertices; the Jaconian set of the *l*-Jaco graph $J_n(x)$, is denoted byJ $(J_n(x))$.Note that the underlying graph will be denoted $J_n^*(x)$ and if the context is clear, both the directed and undirected graph are referred to as a Jaco graph. Similarly the difference between arc and edge and degree, $d_{J_n(x)}(v)$ and $d_{J_n^*(x)}(v)$ will be understood. The Jaco graph has four fundamental properties which are:

(a) $V(J_\infty(x)) = \{v_i : i \in N\}$ and,

(b) If v_j is the head of an arc then the tail is always a vertex $v_i, i < j$ and,

(c) If v_k, for smallest $k \in N$ is a tail vertex then all vertices $v_l, k < l < j$ are tails of arcs to v_j and finally,

(d) The degree of vertex v_k is $d(v_k) = k.$

The family of finite Jaco graph are those limited to $n \in N$ vertices by lobbing off all vertices (and arcs to vertices) $v_t, t > n.$ Hence, trivially $d(v_i) \le i$ for $i \in N$. Figure 1, depicts $J_{10}(x)$.

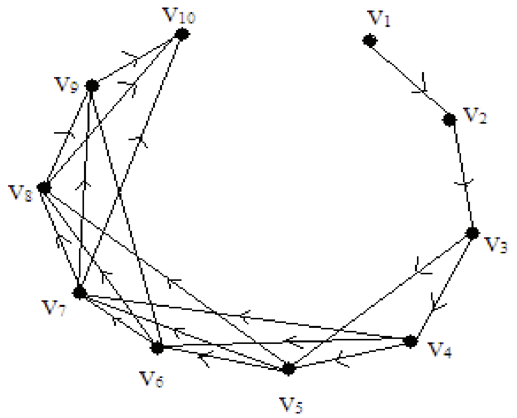


Figure 1: Jaco graph $J_{10}(x)$.

By using Fisher algorithm [4], we can find the degree sequence of $J_n^*(x)$.

For general results and properties refer the paper [4, 6]. These concepts were extended to the quadratic Jaco Graph.

3 Harmonic Polynomial [2]: For an arc (v_i, v_j) of $J_n(x)$ we define the degree pair $(d(v_i), d(v_j))$ in $J_n^*(x)$. Since the vertex v_i has consecutive out-arcs (v_i, v_j), $j \le i + d^+(v_i)$.

Example: For $J_{10}^*(x)$ we have:

(1, 2), (2, 3), (3, 4), (3, 5), (4, 5), (4, 6), (4, 6), (5, 6), (5, 6), (5, 5), (6, 6), (6, 5), (6, 4), (6, 4), (6, 5), (6, 4), (6, 4), (5, 4), (5, 4), (4, 4).

$$H(J_{10}^*(x), x) = 2(x^{1+2-1} + x^{2+3-1} + x^{3+4-1} + x^{3+5-1} + x^{4+5-1} + x^{4+6-1} + x^{4+6-1} + x^{5+6-1} + x^{5+6-1} + x^{5+5-1} + x^{6+6-1} + x^{6+5-1} + x^{6+4-1} + x^{6+4-1} + x^{6+5-1} + x^{6+4-1} + x^{6+4-1} + x^{5+4-1} + x^{5+4-1} + x^{5+4-1} + x^{4+4-1}) = 2(x^2 + x^4 + x^6 + 2x^7 + 3x^8 + 7x^9 + 3x^{10} + 2x^{11}).$$

Also,

$$H(J_{10}(x)) = \int_0^1 2(x^2 + x^4 + x^6 + 2x^7 + 3x^8 + 7x^9 + 3x^{10} + 2x^{11})dx = \frac{11083}{2310}$$

We will use this heuristic method to find the values $1 \le n \le 12$. At this stage both theharmonic polynomial and the harmonic index cannot be determined by anyclosed formula. The heuristic described above is best known at this stage. Finding either a closedformula or an elegant and efficient algorithm is still open. What is observed is the following:For a graph G with $H(G,x) = a_1x^2 + a_2x^3 + a_3x^4 + \ldots + a_nx^{n+1}$. We have, $\varepsilon(G) = \frac{1}{2}\sum_{i=1}^{n} a_i$.

3.1Chromatic Harmonic Polynomial and Chromatic Harmonic Index [5]: We recall that if $C = \{c_1, c_2, \ldots, c_l\}$ is a set of distinct colours [3], a proper vertex colouring of a graph G denoted $\varphi : V(G) \mapsto C$ is a vertex colouring such that no two distinct adjacent vertices have the same colour. The cardinality of a minimum set of colours which allows a proper vertex colouring of G is called the chromatic number of G and is denoted by $\chi(G)$. When a vertex colouring is considered with colours of minimum subscripts the colouring is called a minimum parameter colouring. Unless stated otherwise we consider minimum parameter colour sets throughout this paper. The number of times a colour c_i is allocated to vertices of a graph G is denoted by $\theta(c_i)$ and $\varphi : v_i \mapsto c_j$ is abbreviated, $c(v_i) = c_j$.Furthermore, $c(v_i) = c_j$ then $i(v_i) = j$ .We shall also colour a graph in accordance with the Rainbow Neighborhood Convention [5].

3.2 Rainbow Neighborhood Convention [5] Unless mentioned otherwise we shall consider the colours $C = \{c_1, c_2, \ldots, c_l\}$ and always colour vertices with maximum c_1, followed by maximum $c_{2,\ldots,}$ followed by maximum c_l. Note that the Rainbow Neighborhood Convention ensures a minimum valued chromatic harmonic polynomial and therefore a minimum chromatic harmonic index. The inverse to the convention ensures the maximum valued chromatic harmonic polynomial and the maximum chromaticharmonic index. The inverse colouring requires the mapping $c_j \mapsto c_{l-(j-1)}$ Corresponding to the inverse colouring we define $i'(v_i) = l - (j-1)$ if $c(v_i) = c_j$. In [7] Kok et. al introduced the definitions of the chromatic harmonic polynomials and the chromatic harmonic indices.

Definition 3.1[5]: For a graph G and the minimum parameter colour set $C = \{c_1, c_2, \ldots, c_{\chi(G)}\}$ the minimum (or maximum) chromatic harmonic polynomial $(CHP^- or\ CHP^+)$ and the minimum (or maximum) chromatic harmonic index are $(CHP^- or\ CHP^+)$ defined as:

$$H^{\chi^-}(G,x) = \sum_{v_iv_j \in E(G)} 2x^{i(v_i)+i(v_j)}, \text{ and}$$

$$H^{\chi^-}(G) = \int_0^1 H^{\chi^-}(G,x)$$

$$H^{\chi^+}(G,x) = \sum_{v_iv_j \in E(G)} 2x^{i'(v_i)+i'(v_j)}, \text{ and}$$

$$H^{\chi^+}(G) = \int_0^1 H^{\chi^+}(G,x)$$

3.3 Application to linear Jaco graph:

Theorem 3.3.1: For the *l*-Jaco graph, $J_n(x), n \in N$ with the prime Jaconian vertex v_i we have that the chromatic number, $\chi(J_n^*(x))$ is given by:

$$\chi(J_n^*(x)) = \begin{cases} (n-i)+1, & \text{if and only if the edge } v_iv_n \text{ exists,} \\ n-i, & \text{otherwise.} \end{cases}$$

Applying Theorem 3.3.1 in accordance with the Rainbow Neighbourhood Convention leads to the respective $CHP^-, CHP^+, CHI^- \text{ and } CHI^+$ as depicted in table 1 and 2 below:

Table 1

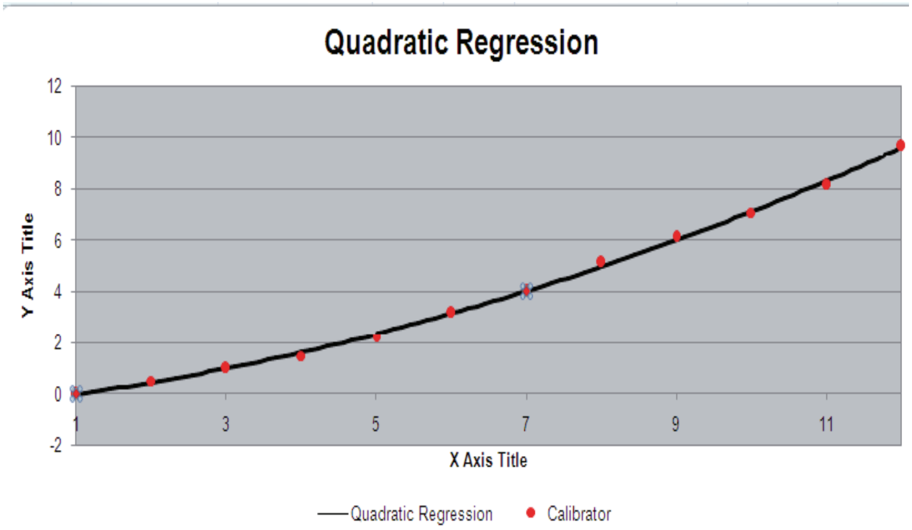
$J_n^*(x)$	$H^{\chi^-}(J_n^*(x),x)$	$H(J_n^*(x))$
1	-	-
2	$2x^3$	$\frac{1}{2} = 0.5$
3	$4x^3$	1
4	$6x^3$	$\frac{3}{2} = 1.5$
5	$6x^3 + 2x^4 + 2x^5$	$\frac{67}{30} \approx 2.2333$
6	$8x^3 + 4x^4 + 2x^5$	$\frac{47}{15} \approx 3.1333$
7	$8x^3 + 4x^4 + 4x^5 + 2x^6 + 2x^7$	$\frac{1681}{420} \approx 4.0024$
8	$10x^3 + 4x^4 + 6x^5 + 4x^6 + 2x^7$	$\frac{717}{140} \approx 5.1214$
9	$10x^3 + 6x^4 + 8x^5 + 4x^6 + 4x^7$	$\frac{641}{105} \approx 6.1048$
10	$10x^3 + 6x^4 + 8x^5 + 6x^6 + 6x^7 + 2x^8 + 2x^9$	$\frac{8899}{1260} \approx 7.0627$
11	$10x^3 + 6x^4 + 10x^5 + 8x^6 + 8x^7 + 2x^8 + 4x^9$	$\frac{5123}{630} \approx 8.1317$
12	$12x^3 + 8x^4 + 12x^5 + 10x^6 + 8x^7 + 2x^8 + 4x^9$	$\frac{608}{63} \approx 9.6508$

Table 2

$J_n^*(x)$	$H^{\chi^+}(J_n^*(x),x)$	$H(J_n^*(x))$
1	-	-
2	$2x^3$	$\frac{1}{2} = 0.5$
3	$4x^3$	1
4	$6x^3$	$\frac{3}{2} = 1.5$
5	$2x^3 + 2x^4 + 6x^5$	$\frac{19}{10} = 1.9$
6	$2x^3 + 4x^4 + 8x^5$	$\frac{79}{30} \approx 2.6333$
7	$2x^3 + 2x^4 + 4x^5 + 4x^6 + 8x^7$	$\frac{659}{210} \approx 3.1381$
8	$2x^3 + 4x^4 + 6x^5 + 4x + 10x^7$	$\frac{577}{140} \approx 4.1215$
9	$10x^3 + 6x^4 + 8x^5 + 4x^6 + 4x^7$	$\frac{2201}{420} \approx 5.2405$
10	$2x^3 + 2x^4 + 6x^5 + 6x^6 + 8x^7 + 6x^8 + 10x^9$	$\frac{1139}{210} \approx 5.4238$
11	$4x^3 + 2x^4 + 8x^5 + 8x^6 + 10x^7 + 6x^8 + 10x^9$	$\frac{951}{140} \approx 6.7929$
12	$4x^3 + 2x^4 + 8x^5 + 10x^6 + 12x^7 + 8x^8 + 12x^9$	$\frac{4883}{630} \approx 7.7508$

In respect of table 2, the quadratic best curve fit found to approximate the $H^{\chi^-}(J_n^*(x))$ index For $n \in N$ is given by

Coefficient	Value	$\pm$ Error
a	-0.4117045	0.1124476
b	0.03465165	0.0397701
c	0.04054241	0.0029781

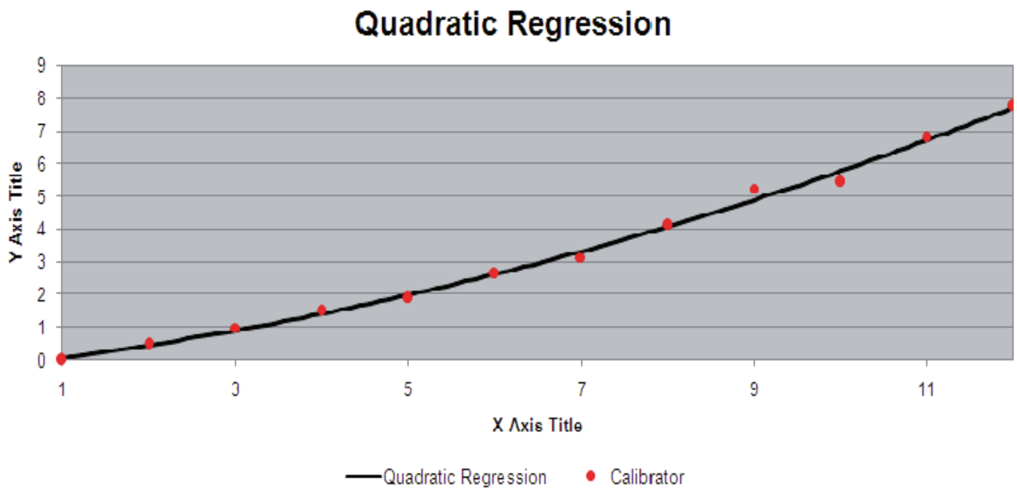


X-axis	Y-axis
1	0
2	0.5
3	1
4	1.5
5	2.2333
6	3.1333
7	4.0024
8	5.1214
9	6.1048
10	7.0627
11	8.1317
12	9.6508

$$H^{\chi^-}(J_n^*(x)) = -0.4117045 + 0.3465165n + 0.04054241n^2$$

Similarly, for the $H^{\chi^+}(J_n^*(x))$ index for $n \in N$ is given by

Coefficient	Value	$\pm$ Error
a	-0.2451568	0.1932342
b	0.29282849	0.0683425
c	0.03092639	0.0051177



X-axis	Y-axis
1	0
2	0.5
3	1
4	1.5
5	1.9
6	2.6333
7	3.1381
8	4.1215
9	5.2405
10	5.4238
11	6.7929
12	7.7508

$$H^{\chi^-}(J_n^*(x)) = 0.0309264\,n^2 + 0.2928285\,n - 0.2451568$$

Conclusion: In this paper, we have reviewed the concept of Linear Jaco graph. It has been extended to the Quadratic Jaco graph. Also we have found the Harmonic polynomial and its respective Index. In linear Jaco Graph, we have found the Chromatic Harmonic Polynomial and Chromatic Harmonic Index for minimum and maximum n, $2 \le n \le 12$. By using statistical method, the best fit for $H^{\chi^-}(J_n^*(x))$ and $H^{\chi^-}(J_n^*(x))$ have been found. These can be extended to find the properties in the general form.

References:

1. K.Sujatha, M.Chandramouleeswaran, Intuitionistic L- Fuzzy B-Filters On B-Algebras; Mathematical Sciences International Research Journal : ISSN 2278-8697Volume 4 Issue 2 (2015), Pg 329-331
2. J.A. Bondy, U.S.R. Murty, GraphTheory with Applications, Macmillan Press, London, (1976).
3. S.S.Benchalli , P.G.Patil , N.S.Kabbur, On Soft Γ-Connected Spaces In Soft Topological Spaces; Mathematical Sciences International Research Journal : ISSN 2278-8697Volume 4 Issue 2 (2015), Pg 332-335
4. M. A. Iranmanesh, M. Saheli, "On the harmonic index and harmonic polynomial of Catepillars with diameter four", Iranian Journal of Mathematical Chemistry, 5 (2)
5. (2014): 35-43.
6. Priti Gupta,Prince,Vijay Kumar, Comparative Analysis Between intuitionistic Fuzzy Set; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 3 Issue 1 (2014), Pg 436-440
7. J. Kok, N.K. Sudev and K.P. Chithra, "General colouring sums of graphs", Cogent Mathematics,(2016), 3, 1140002.
8. J. Kok, C. Susanth, S.J. Kalayathankal, "A Study on Linear Jaco Graphs", Journal of
9. S.Sripriya,Ajeet Singh,B.L.Raina, Zeros of Polynomials With Extremecoefficients; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 3 Issue 1 (2014), Pg 432-435
10. Informaticsand Mathematical Sciences, 7(2) (2016): 129-135.9
11. J. Kok and K.A. Germina, "Chromatic Harmonic Indices and Chromatic Harmonic Polynomialsof Certain Graphs", Communicated.
12. Pankaj Thakur, Anupam Semwal, Steady State thermal Stresses in Thin Circular Disc; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 3 Issue 1 (2014), Pg 421-431
13. U. Mary, M. Jerlin Seles, R. Jayasree, "Zagreb Indices of the Thorn Jaco Graph",
14. Dhounsi, K.S, Yasmeen, on Generalized Hermite Polynomials of Three Variables; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 4 Issue 1 (2015), Pg 185-187
15. Journal ofInformatics and Mathematical Sciences, 7(2) (2015): 69-80.
16. L. Zhong, "The harmonic index for graphs", Applied Mathematics Letters, 25 (2012): 561-566.

R. Jayasree
Research Scholar, Department of Mathematics, Nirmala College for Women, Coimbatore, India
A. Kulandai Therese, U. Mary
Assistant professor, Department of Mathematics, Nirmala College for Women, Coimbatore, India
Johan Kok
Tshwane Metropolitan Police Department, City of Tshwane, Republic of South Africa.