k-EVEN SEQUENTIAL HARMONIOUS LABELING OF SOME SPECIAL GRAPHS

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Abstract: Graham and Sloane [7] introduced the harmonious graphs and Singh & Varkey [11] introduced the odd sequential graphs. Gayathri and Hemalatha [2] introduced even sequential harmonious labeling of graphs. We have introduced *k*-even sequential harmonious labeling of graphs. In this paper, we investigate some properties, necessary conditions for *k*-even sequential harmonious labeling. Also, we present *k*-even sequential harmonious labeling schemes for some special graphs such as split graphs and shadow graphs. Further, we prove that K_6 is not an even sequential harmonious graph.

Keywords: Harmonious graphs, *k*-even sequential harmonious labeling, *k*-even sequential harmonious labeling graphs.

1. Introduction: All graphs in this paper are finite, simple and undirected. The symbols $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph *G*. The cardinality of the vertex set is called the **order** of *G*. The cardinality of the edge set is called the **size** of *G*. A graph with *p* vertices and *q* edges is called a **(*p*, *q*) graph**.
Graham and Sloane [2] introduced the harmonious graphs and Singh & Varkey [6] introduced the odd sequential graphs. Harmonious and related graphs are dealt in [3-5]. We refer to the excellent survey by Gallian [1] for varieties of labeling and graphs.
Gayathri and Hemalatha [1] say that a labeling is an **even sequential harmonious labeling** if there exists an injection *f* from the vertex set *V* to {0, 1, 2, ..., 2q} such that the induced mapping f^+ from the edge set *E* to {2, 4, 6, ..., 2*q*} defined by $f^+(uv) = \begin{cases} f(u)+f(v), \text{if } f(u)+f(v) \text{ is even} \\ f(u)+f(v)+1, \text{if } f(u)+f(v) \text{ is odd} \end{cases}$ are distinct.
A graph *G* is said to be an **even sequential harmonious graph** if it admits an even sequential harmonious labeling. We have introduced *k*-even sequential harmonious labeling by extending the above definition for any integer $k \geq 1$. We say that a labeling is a ***k*-even sequential harmonious labeling** if there exists an injection *f* from the vertex set *V* to $\{k-1, k, k+1, ..., k+2q-1\}$ such that the induced mapping f^+ from the edge set *E* to $\{2k, 2k+2, 2k+4, ..., 2k+2q-2\}$ defined by $f^+(uv) = \begin{cases} f(u)+f(v), \text{if } f(u)+f(v) \text{ is even} \\ f(u)+f(v)+1, \text{if } f(u)+f(v) \text{ is odd} \end{cases}$ are distinct.
A graph *G* is said to be a ***k*-even sequential harmonious graph** if it admits an *k*-even sequential harmonious labeling. Throughout this paper, *k* denote any positive integer greater than or equal to 1. For brevity, we use *k*-ESHL for *k*-even sequential harmonious labeling and *k*-ESHG for *k*-even sequential Harmonious graph.
In this paper, we investigate some properties, necessary conditions for *k*-even sequential harmonious labeling. Also, we present *k*-even sequential harmonious labeling schemes for some special graphs such as split graphs, shadow graphs. Further, we prove that K_6 is not an even sequential harmonious graph.

2. Properties and Necessary conditions of *k*-ESHL

Theorem 2.1: The total number of possible *k*-ESHL of a graph with *q* edges is $2^q(q!)$.

Proof

The set of labeling for the edges $\{2k, 2k+2, 2k+4, ..., 2k+2q-2\}$ given as follows.

Edge Label	Choice of labels for the adjacent vertices	No. of Possibilities
$2k$	$(k-1, k)$ and $(k-1, k+1)$	2
$2k+2$	$(k-1, k+3)$, $(k-1, k+2)$, $(k, k+2)$, $(k, k+1)$	4
$2k+4$	$(k-1, k+5)$ $(k-1, k+4)$, $(k, k+4)$ $(k, k+3)$, $(k+1, k+3)$ $(k+1, k+2)$	6
$2k+6$	$(k-1, k+7)$ $(k-1, k+6)$, $(k, k+6)$, $(k, k+5)$, $(k+1, k+5)$ $(k+1, k+4)$, $(k+2, k+4)$ $(k+2, k+3)$	8
...	...	...
$2k+2q-2$	$(k-1, k+2q-1)$, $(k-1, k+2q-2)$, $(k, k+2q-2)$, $(k, k+2q-3)$, $(k+1, k+2q-3)$ $(k+1, k+2q-4)$	2q
	$(k+q-2, k+q)$, $(k+q-2, k+q-1)$	

Hence total number of possible *k*-even sequential harmonious labeling = 2.4.6.8….2*q* = (2.2.2….2) (1.2.3… *q*) = $2^q(q!)$.

Observation 2.2

1. If G is a k-*ESHG* then $k - 1$ has to be a vertex label.
2. If $G = (p, q)$ is a k-*ESHG* then either $(k - 1, k)$ or $(k - 1, k + 1)$ must appear as adjacent vertex labels.
3. If $G = (p, q)$ is a 1-*ESHG* then G is k-*ESHG* for any $k > 1$.

Theorem 2.3: Let G be a even sequential harmonious graph with even sequential harmonious labeling f. Let t be the number of edges whose one vertex label is even and the other is odd. Then $\sum_{v\in V(G)} d(v)f(v)+t=q[2k+q-1]$ where $d(v)$ denotes the degree of a vertex v.

Proof: By the definition of even sequential harmonious labeling,

$$f^+(xy)=\begin{cases} f(x)+f(y) & \text{if } f(x)+f(y) \text{ is even} \\ f(x)+f(y)+1 & \text{if } f(x)+f(y) \text{ is odd}\end{cases}$$

$\sum d(v)f(v)=\left[\sum f^+(xy)-t\right]$ $= [2k + 2k + 2 + \ldots + 2k + 2q - 2] - t$

$= [2qk + 2(1 + \ldots + q - 1)] - t$

$= 2qk + \frac{2(q-1)(q)}{2} - t = q[2k + q - 1] - t$

$$\therefore \sum_{v\in V} d(v)f(v)+t=q[2k+q-1]$$

Corollary 2.4: If G is a k-even sequential harmonious graph with k-even sequential harmonious labeling f, then $\sum d(v)f(v)\ge q^2$.

Proof: From theorem 2.3,

$\sum d(v)f(v) = q[2k + q - 1] - t \ge 2qk + q^2 - 2q$ (since $t \le q$)

$= q^2 + 2q(k - 1)$

As $k \ge 1$, we have $\sum d(v)f(v)\ge q^2$.

Theorem 2.5: Let G be a k-even sequential harmonious graph containing a cycle $C_3 = uvwu$. If f is a k-even sequential harmonious labeling of G, then $\{f(u), f(v), f(w)\} \neq \{k - 1, k, k + 1\}$ and $\{k, k + 1, k + 2\}$ for any k.

Proof:

Suppose $\{f(u), f(v), f(w)\} = \{k - 1, k, k + 1\}$. Assume $f(u) = k - 1$, $f(v) = k$, $f(w) = k + 1$. Then the edges uv and uw get the same label $2k$. This is a contradiction.

Suppose $\{f(u), f(v), f(w)\} = \{k, k + 1, k + 2\}$. Assume $f(u) = k$, $f(v) = k + 1$, $f(w) = k + 2$. Then the edges uw and uv get the same label $2k + 2$. This is a contradiction.

Theorem 2.6: In every k-even sequential harmonious graph, $k - 1$ has to be one of the vertex labels.

Proof: The edge label $2k$ is possible only when the adjacent vertex labels are of the form $(k - 1, k)$ or $(k - 1, k + 1)$. Thus the vertex label $k - 1$ is mandatory.

Corollary 2.7: In every 1-even sequential harmonious graph, 0 has to be one of the vertex labels.

Proof: Follows from Theorem 2.6 and $k = 1$.

Theorem 2.8: Let $G = (p, q)$ be a l-regular 1-even sequential harmonious graph with l even and t be the number of edges whose one vertex label is even and other is odd. Then t is even.

Proof:

From Theorem 2.3, we have $\sum d(v)f(v)+t = q(q + 1)$.

G being a l-regular graph, $l\sum f(v)+t = q(q + 1)$.

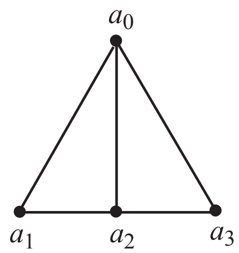
$\Rightarrow t = q(q + 1) - l\sum f(v)$ which is even, as l is even. t is even.

Theorem 2.9: Let $G = (p, q)$ be a even sequential harmonious graph. If l is the label of some edge uv and $f(u) = x$, $f(v) = y$ then, $0 \le x \le \frac{l}{2}-1$, $\frac{q}{2} \le y \le l$

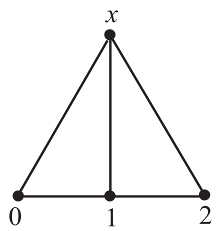
Proof: Follows from Theorem 2.1 with $k = 1$.

Theorem 2.10: In any even sequential harmonious graph $G = (p, q)$, 0, 1, 2 cannot be the labels of the same F_3 contained in G.

Proof: Suppose not, let a_0, a_1, a_2, a_3 be the vertices of F_3 in G.
Then the following cases arise



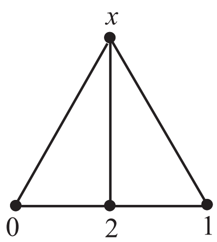
Case 1: $f(a_0) = x; f(a_1) = 0; f(a_2) = 1, f(a_3) = 2$



Subcase 1: *x* is odd: In this case, the induced edge labels $f^+(a_0a_1) = x+1$ and $f^+(a_0a_2) = x+1$ are equal, $a \Rightarrow \Leftarrow$.

Subcase 2: *x* even: In this case, the induced edge labels $f^+(a_0a_2) = x+2$, $f^+(a_0a_3) = x+2$ are equal, $a \Rightarrow \Leftarrow$.

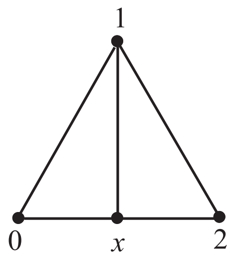
Case 2: $f(a_0) = x; f(a_1) = 0; f(a_2) = 2; f(a_3) = 1$



Subcase 1: *x* odd: In this case, the induced edge labels $f^+(a_0a_1) = x+1 = f^+(a_0a_3)$, $a \Rightarrow \Leftarrow$.

Subcase 2: *x* even: In this case, the induced edge labels $f^+(a_0a_2) = x+2 = f^+(a_0a_3)$, $a \Rightarrow \Leftarrow$.

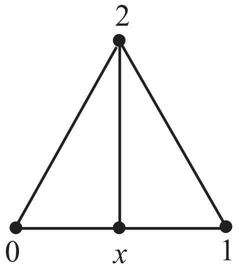
Case 3: $f(a_0) = 1; f(a_1) = 0; f(a_2) = x; f(a_3) = 2.$



Subcase 1: *x* odd: In this case, the induced edge labels $f^+(a_0a_2) = x+1 = f^+(a_1a_2)$, $a \Rightarrow \Leftarrow$.

Subcase 2: *x* even: In this case, the induced edge labels $f^+(a_0a_2) = x+2 = f^+(a_2a_3)$, $a \Rightarrow \Leftarrow$.

Case 4: $f(a_0) = 2; f(a_1) = 0; f(a_2) = x; f(a_3) = 1.$



Subcase 1: x odd: In this case, the induced edge labels $f^+(a_1a_2) = x+1 = f^+(a_2a_3)$, $a \Rightarrow \Leftarrow$.

Subcase 2: x is even: In this case, the induced edge labels $f^+(a_0a_2) = x+2 = f^+(a_2a_3)$, $a \Rightarrow \Leftarrow$.

Hence the theorem.

Theorem 2.11: If the graph $G = (p, q)$ is a 1-even sequential harmonious graph with no pendant vertices then $2q$ and $2q - 1$ cannot be the labels of the vertices.

Proof: For a 1-*ESHG*, 0 has to be the vertex label. Suppose $2q$ is a vertex label. Let y be the vertex with $f(v) = 2q$. Since y is not a pendant vertex there are atleast two vertices u and v adjacent to y. Also $f(u)$, $f(v) \geq 0$. Let $f(u) = 0$ and $f(v) > 0$. Then $f^+(yv) > 2q$, $a \Rightarrow \Leftarrow$. Similar argument holds for $2q - 1$.

Theorem 2.12: Let $G = (p, q)$ be a 1-even sequential harmonious graph with maximum degree Δ. Let f be a 1-even sequential harmonious labeling of G. Then $f(u) \leq 2q - 2\Delta + 3$ for every u.

Theorem 2.13: Let $G = (p, q)$ be a 1-even sequential harmonious graph. Then any vertex v with label $f(v) \in \{0, 1, 2, ..., 2q\}$ has following relation.

$$d(v) \leq \begin{cases} q & \text{if } f(v) = 0 \\ \dfrac{2[q+1] - f(v)}{2} & \text{if } f(v) \neq 0 \,\&\, f(v) \text{ is even} \\ \dfrac{2q+1-f(v)}{2} & \text{if } f(v) \neq 0 \,\&\, f(v) \text{ is odd} \end{cases}$$

Proof

Subcase 1: $f(v) = 0$

If $f(v) = 0$ then its adjacent vertex labels are from the set $\{1, 3, 5, ..., 2q - 1\}$ ... (*) or from the set $\{2, 4, 6, ..., 2q\}$ .... (**).

Hence, the total number of possible adjacent vertex labels is =

$$\frac{2q-1-1}{2}+1=\frac{2q}{2}=q \quad \text{(from *) or} \quad \frac{2q-2}{2}+1=\frac{2q}{2}=q \quad \text{(from **)}$$

Therefore if $f(v) = 0$ then the total number of adjacent vertex labels of v is q.

Subcase 2: $f(v) \neq 0$ & $f(v)$ is odd

Let $f(v) = 2l - 1$ .... (***) say. Then its maximum possible adjacent vertex labels are from the set $\{0, 2, 4, 6, ... 2(q - l)\}$ .... (****)

Hence, the total number of possible adjacent vertex labels is

$$=\frac{2q-2l-0}{2}+1=\frac{2q-2l+2}{2} \quad \text{(from ****)}$$

$$=\frac{2q-(f(v)+1)+2}{2} \quad \text{(from ***)}$$

$$=\frac{2q-f(v)+1}{2}$$

Therefore if $f(v) \neq 0$ and $f(v)$ is odd then the total number of possible adjacent vertex labels is $\dfrac{2q - f(v)+1}{2}$.

Subcase 3: $f(v) \neq 0, 2q$ and $f(v)$ is even:

Let $f(v) = 2l$ say .... (*****)

Then its maximum possible adjacent vertex labels are from the set $\{1, 3, 5, \ldots, 2q - 2l - 1\} \cup \{0\}$.

Hence, the total possible vertex labels is $= \dfrac{2q-2l-1-1}{2} + 1 + 1 = \dfrac{2q-2l+2}{2}$

$= \dfrac{2q - f(v)+2}{2}$ (from *****)

Hence the theorem.

Subcase 4:

f(v) = 2q: Then the only possible adjacent vertex label is 0. Hence the total possible vertex label is 1.

Corollary 2.14: Let $G = (p, q)$ be a 1-*ESHG*. Let x be a vertex with label $f(v)$ then,

$$d(v) \leq \begin{cases} q & \text{if } f(v)=0,1,2 \\ q-1 & \text{if } f(v)=3,4 \\ q-2 & \text{if } f(v)=5,6 \\ \ldots\ldots. & \ldots\ldots \\ 2 & \text{if } f(v)=2q-3, 2q-2 \\ 1 & \text{if } f(v)=2q-1, 2q \end{cases}$$

Theorem 2.15: The complete graph K_6 is not a 1-even sequential harmonious graph.

Proof: Let the vertices and edges of K_6 be as given in Fig. 2.1.

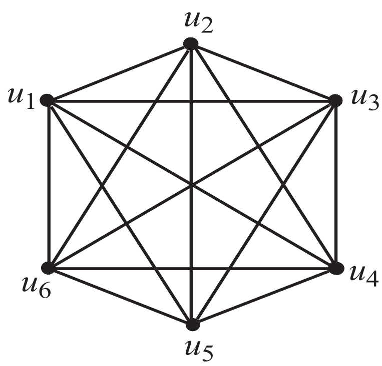


Fig. 2.1

Suppose K_6 is a 1-ESHG then there exists a function $f : V(K_6) \to \{0, 1, 2, \ldots, 2q = 30\}$ such that $f^+(E(K_6)) = \{2, 4, 6, 8, \ldots, 2q\}$. Here $q = 15$ & $2q = 30$.

Claim 1: There is atmost one vertex with label greater than 15.: On the contrary, suppose there are two vertices say u_5 and u_4 with label > 15. Without loss of generality, assume that $f(u_5) = l_1$, $f(u_4) = l_2$ and $l_1, l_2 > 15$.

Therefore $f^+(u_4u_5) = l_1 + l_2 > 30$, $a \Rightarrow \Leftarrow$. Hence claim 1.

Claim 2: $f(u_i) \ngeq 23\ \forall\ i$:

Suppose $f(u_i) \geq 23$ for some i say $i = 3$ (i.e.) $f(u_3) = 23$.

Then by previous theorem, $d(u_3) \leq \dfrac{2q+1-f(u_3)}{2} = \dfrac{30+1-23}{2} = 4$

Therefore u_3 can have atmost 4 adjacent vertices. But $deg(u_3) = 5$ implies that it is not possible to assign any value to the 5^{th} vertex adjacent to u_3.
Similar argument holds when $f(u_i) > 23$. Hence claim 2.
From claim 1 and 2, it follows that there is only one vertex
with label belongs to {16, 17, ..., 22} .... (*)
To receive the edge label 2, the possibilities of adjacent vertex labels are (0, 1) or (0, 2)
Case 1: 0 and 1 are adjacent vertex labels: Let $f(u_1) = 0, f(u_2) = 1$. The graph being complete even sequential harmonious, it is clear that none of the vertices of $\{u_3, u_4, u_5, u_6\}$ can take odd values.

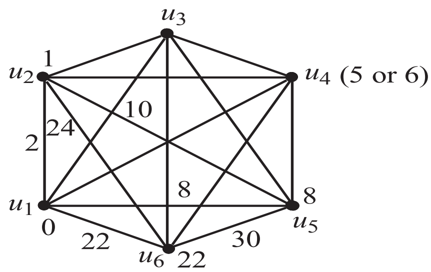
To receive the edge label 30, the possible pairs (from *) are

a) (22, 7)	b) (22, 8)	c) (21, 8)	d) (21, 9)	
e) (20, 9)	f) (20, 10)	g) (19, 10)	h) (19, 11)	... (**)
i) (18, 11)	j) (18, 12)	k) (17, 12)	l) (17, 13)	
m) (16, 13)	n) (16, 14)	o) (15, 14)		

By omitting odd values for u_i, i = 3, 4, 5, 6 we get the possibilities as
a_1) (22, 8) a_2) (20, 10) a_3) (18, 12) a_4) (16, 14)

Case a_1: (22, 8) are vertex labels

Let $f(u_6) = 22, f(u_5) = 8$



Since $f(u_6) = 22$, $f(u_i) \leq 7$ for i = 3, 4.

To receive the edge label 28, we must have 5 or 6 as vertex label. Let $f(u_4)$ = 5 or 6.

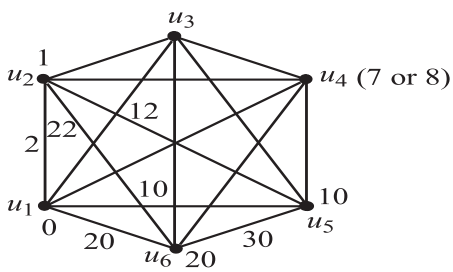
Suppose $f(u_4) = 6$, then $f^+(u_2u_4) = 8 = f^+(u_1u_5)$, $a \Rightarrow \Leftarrow$.

Suppose $f(u_4) = 5$, then $f^+(u_1u_4) = 6 = f^+(u_2u_4)$, $a \Rightarrow \Leftarrow$.

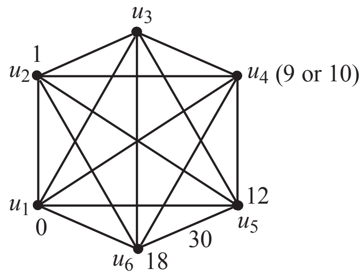
Therefore 28 cannot be an edge label.

Case a_2: (20, 10) are vertex labels

Let $f(u_6) = 20$, $f(u_5) = 10$



Since $f(u_6) = 20$, $f(u_i) \leq 9\ \forall\ i$ = 3, 4.
To receive the edge label 28, we must have 7 or 8 as vertex label. Let $f(u_4)$ = 7 or 8.
Suppose $f(u_4) = 7$, then $f^+(u_2u_4) = 8 = f^+(u_1u_4)$, $a \Rightarrow \Leftarrow$.
Suppose $f(u_4) = 8$, then $f^+(u_2u_4) = 10 = f^+(u_1u_5)$, $a \Rightarrow \Leftarrow$.
Case a_3: (18, 12) are vertex labels: Let $f(u_6) = 18$, $f(u_5) = 12$



Since $f(u_6) = 18$, $f(u_i) \leq 11$ for $i = 3, 4$.

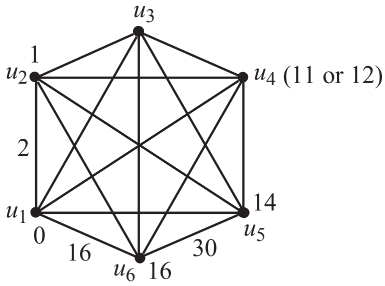
To receive the edge label 28, we must have 9 or 10 as vertex label. Let $f(u_4) = 9$ or 10.

Suppose $f(u_4) = 9$, then $f^+(u_2u_4) = 10 = f^+(u_1u_4)$, $a \Rightarrow\Leftarrow$.

Suppose $f(u_4) = 10$, then $f^+(u_2u_4) = 12 = f^+(u_1u_5)$, $a \Rightarrow\Leftarrow$.

Case a_4: (16, 14) are vertex labels

Let $f(u_6) = 16$, $f(u_5) = 14$.



Since $f(u_6) = 16$, $f(u_i) \leq 12$ for $i = 3, 4$.

To receive the edge label 28, we must have 11 or 12 as vertex label. Let $f(u_4) = 11$ or 12.

Suppose $f(u_4) = 11$, then $f^+(u_2u_4) = 12 = f^+(u_1u_4)$, $a \Rightarrow\Leftarrow$.

Suppose $f(u_4) = 12$, then $f^+(u_2u_4) = 14 = f^+(u_1u_5)$, $a \Rightarrow\Leftarrow$.

Case 2: 0 and 2 are adjacent vertex labels: Let $f(u_1) = 0$, $f(u_2) = 2$.

To receive the edge label 30, we have the adjacent vertex labels as noted in (**) of case 1.

Then one can arrive at the repeated labels in each of the subcases of (**) and hence, it follows that K_6 is not a 1-*ESHG*.

3. *k*-even sequential harmonious labeling schemes:

Definition 3.1: The **shadow graph** $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbours of the corresponding vertex v' in G''.

Case 2: 0 and 2 are adjacent vertex labels: Let $f(u_1) = 0$, $f(u_2) = 2$.

Definition 3.2: For a graph G the **split graph** is obtained by adding to each vertex v a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G. The resultant graph is denoted as $spl(G)$.

Theorem 3.3: $D_2(P_n)$ $(n \geq 3)$ is a k-even sequential harmonious graph for any k.

Proof: Let $\{v_1, v_2, ..., v_n, v_1', v_2', ..., v_n'\}$ be the vertices and the set $\{e_i, e_i', 1 \leq i \leq n-1, a_i, 1 \leq i \leq 2n-2\}$ be the edges which are denoted as in Fig. 3.1.

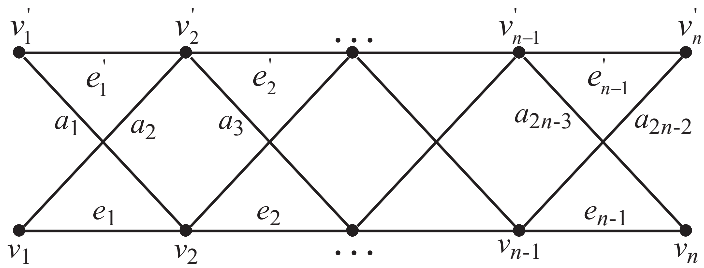


Fig. 3.1: $D_2(P_n)$ with ordinary labeling

First we label the vertices as follows:

Define $f : V \rightarrow \{k, k-1, k+1, ..., k+2q-1\}$ by

Case (i): *n* is even

$f(v_1) = k - 1;$ $f(v_2) = k$

For $1 \le i \le \frac{n-2}{2}$, $f(v_{2i+1}) = k + 8i - 1$

$f(v_{2i+2}) = k + 8i;$ $f(v_1') = k + 3;$ $f(v_2') = k + 2$

For $1 \le i \le \frac{n-2}{2}$, $f(v_{2i+1}') = k + 8(i - 1) + 11;$ $f(v_{2i+2}') = k + 8(i - 1) + 10$

Then the induced edge labels are:

For $1 \le i \le n - 1$, $f^+(e_i) = 2k + 8(i - 1)$

$f^+(e_i') = 2k + 8i - 2;$ $f^+(a_1) = 2k + 2$

$f^+(a_2) = 2k + 4;$ $f^+(a_3) = 2k + 10$

For $4 \le i \le 2n - 2$,

$$f^+(a_i) = \begin{cases} 2k + 4i - 6 & i \equiv 2 \pmod 4 \\ 2k + 4i - 2 & i \equiv 3 \pmod 4 \end{cases}$$

$$f^+(a_i) = \begin{cases} 2k + 4i - 4 & i \equiv 0 \pmod 4 \\ 2k + 4i & i \equiv 1 \pmod 4 \end{cases}$$

Case (ii): *n* is odd:

$f(v_1) = k - 1;$ $f(v_2) = k$

For $1 \le i \le \frac{n-1}{2}$, $f(v_{2i+1}) = k + 8i - 1$

For $1 \le i \le \frac{n-3}{2}$, $f(v_{2i+2}) = k + 8i$

$f(v_1') = k + 3;$ $f(v_2') = k + 2$

For $1 \le i \le \frac{n-1}{2}$, $f(v_{2i+1}') = k + 8i + 3$

For $1 \le i \le \frac{n-3}{2}$, $f(v_{2i+2}') = k + 8i + 2$

Then the induced edge labels are:

For $1 \le i \le n - 1$, $f^+(e_i) = 2k + 8(i - 1);$ $f^+(e_i') = 2k + 8i - 2$

$f^+(a_1) = 2k + 2;$ $f^+(a_2) = 2k + 4;$ $f^+(a_3) = 2k + 10$

For $4 \le i \le 2n - 2$,

$$f^+(a_i) = \begin{cases} 2k + 4i - 6 & i \equiv 2 \pmod 4 \\ 2k + 4i - 2 & i \equiv 3 \pmod 4 \end{cases}$$

$$f^+(a_i) = \begin{cases} 2k + 4i - 4 & i \equiv 0 \pmod 4 \\ 2k + 4i & i \equiv 1 \pmod 4 \end{cases}$$

Therefore, $f^+(E) = \{2k, 2k + 2, ..., 2k + 2q - 2\}$. So, *f* is a *k*-even sequential harmonious labeling and hence, $D_2(P_n)$ $(n \ge 3)$ is a *k*-even sequential harmonious graph for any *k*.

Illustration 3.4: 2-*ESHL* of $D_2(P_8)$ is shown in Fig. 3.2.

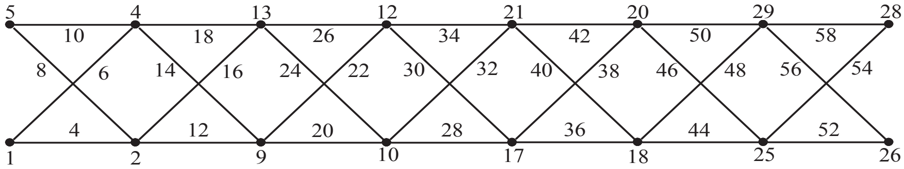


Fig. 3.2: 2-*ESHL* of $D_2(P_8)$

Theorem 3.5: $D_2(K_{1,n})$ is a k-even sequential harmonious graph for any k.

Proof: Let $\{v_1, v_2, ..., v_n, v_1', v_2', ..., v_n', v, v'\}$ be the vertices and $\{a_i, b_i, 1 \le i \le n, a_i', b_i', 1 \le i \le n\}$ be the edges which are denoted as in Fig. 3.3.

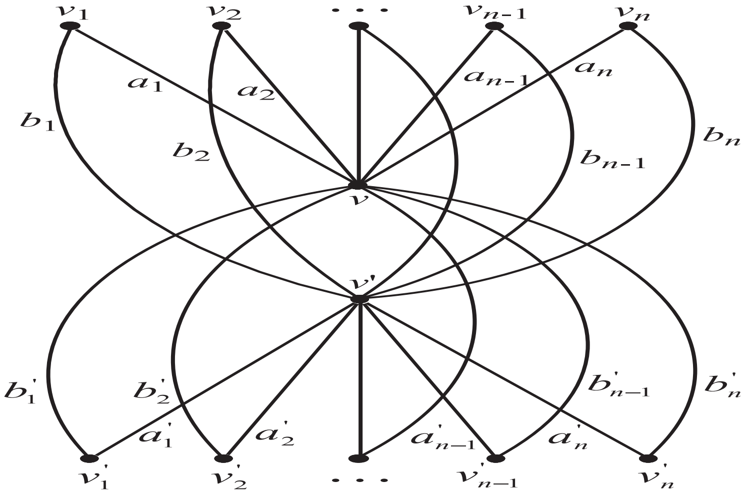


Fig. 3.3: $D_2(K_{1,n})$ with ordinary labeling

First we label the vertices as follows:

Define $f : V \to \{k, k-1, k+1, ..., k+2q-1\}$ by $f(v) = k-1$

For $1 \le i \le n$, $f(v_i) = k + 4i - 4$; $f(v') = k + 1$

For $1 \le i \le n$, $f(v_i') = k + 4n + 4(i-1)$

Then the induced edge labels are:

For $1 \le i \le n$, $f^+(a_i) = 2k + 4(i-1)$; $f^+(b_i) = 2k + 4i - 2$

$f^+(a_i') = 2k + 4(n+i) - 2$; $f^+(b_i') = 2k + 4n + 4(i-1)$

Therefore, $f^+(E) = \{2k, 2k+2, ..., 2k+2q-2\}$. So, f is a k-even sequential harmonious labeling and hence, $D_2(K_{1,n})$ is a k-even sequential harmonious graph for any k.

Illustration 3.6: 4-ESHL of $D_2(K_{1,6})$ is shown in Fig. 3.4.

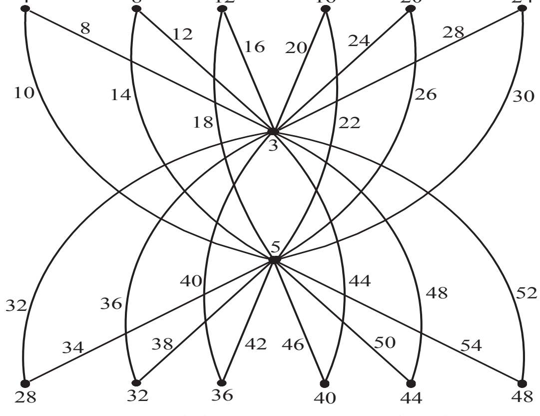


Fig. 3.4(b): 4-ESHL of $D_2(K_{1,6})$

Theorem 3.7: $spl(P_n)$ $(n \geq 2)$ is a k-even sequential harmonious graph for any k.
Proof: Let $\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ be the vertices and $\{e_i, 1 \leq i \leq n-1, a_i, 1 \leq i \leq 2n-2\}$ be the edges which are denoted as in Fig. 3.5.

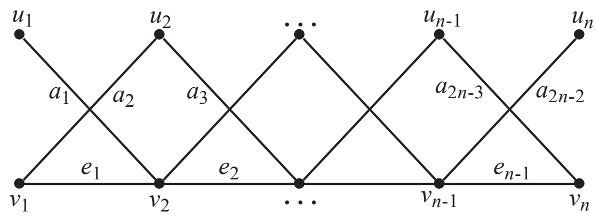


Fig. 3.5: *spl*(P_n) with ordinary labeling

First we label the vertices as follows:
Define $f: V \to \{k, k-1, k+1, \ldots, k+2q-1\}$ by

For $1 \leq i \leq n$, $f(v_i) = \begin{cases} k+3i-4 & i \text{ odd} \\ k+3i-6 & i \text{ even} \end{cases}$

For $1 \leq i \leq n$, $f(u_i) = k + 3i - 2$

Then the induced edge labels are: For $1 \leq i \leq n-1$, $f^+(e_i) = 2k + 6(i-1)$

For $1 \leq i \leq 2n-2$,

$$f^+(a_i) = \begin{cases} 2k+3i+1 & i \equiv 3 \pmod 4 \\ 2k+3i-2 & i \equiv 2 \pmod 4 \end{cases}$$

$$f^+(a_i) = \begin{cases} 2k+3i-1 & i \equiv 1 \pmod 4 \\ 2k+3i-4 & i \equiv 0 \pmod 4 \end{cases}$$

Therefore, $f^+(E) = \{2k, 2k+2, \ldots, 2k+2q-2\}$. So, f is a k-even sequential harmonious labeling and hence, $spl(K_{1,n})$ $(n \geq 2)$ is a k-even sequential harmonious graph for any k.
Illustration 3.8: 7-ESHL of $spl(P_7)$ is shown in Fig. 3.6.

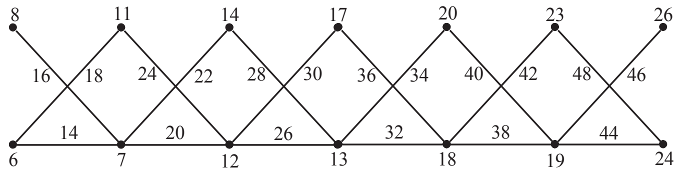


Fig. 3.6: 7-ESHL of *spl*(P_7)

Theorem 3.9: $spl(K_{1,n})$ $(n \geq 2)$ is a k-even sequential harmonious graph for any k.

Proof: Let $\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n, v, u\}$ be the vertices and $\{a_i, a_i', b_i, 1 \leq i \leq n\}$ be the edges which are denoted as in Fig. 3.7.

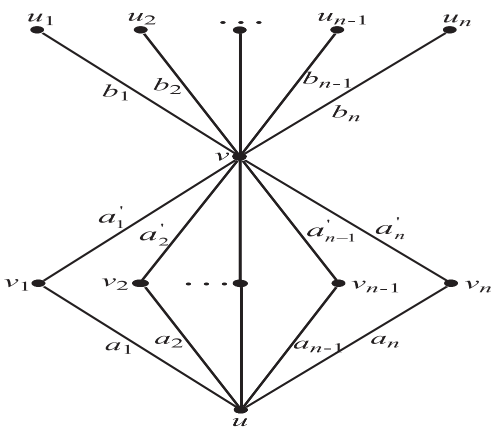


Fig. 3.7: $spl(K_{1,n})$ with ordinary labeling

First we label the vertices as follows:

Define $f: V \rightarrow \{k, k-1, k+1, ..., k+2q-1\}$ by $\quad f(v) = k - 1$

For $1 \le i \le n$, $\quad f(v_i) = k + 2n + 4(i-1)$; $\quad f(u) = k + 1$

For $1 \le i \le n$, $\quad f(u_i) = k + 2i - 2$

Then the induced edge labels are:

For $1 \le i \le n$, $\quad f^+(a_i) = 2k + 2n + 4i - 2$; $\quad f^+(a_i^{'}) = 2k + 2n + 4(i-1)$

$f^+(b_i) = 2k + 2(i-1)$

Therefore, $f^+(E) = \{2k, 2k+2, ..., 2k+2q-2\}$. So, f is a k-even sequential harmonious labeling and hence, $spl(K_{1,n})$ $(n \ge 2)$ is a k-even sequential harmonious graph for any k.

Illustration 3.10: 4-ESHL of $spl(K_{1,7})$ is shown in Fig. 3.8.

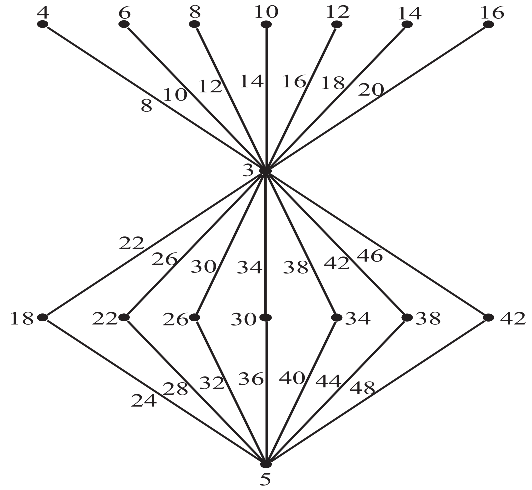


Fig. 3.8: 4-*ESHL* of $spl(K_{1,7})$

Theorem 3.11: $D_2(B_{n,n})$ $(n \ge 3)$ is a k-even sequential harmonious graph for any k.

Proof: Let $\{u, v, v_i, u_i, 1 \le i \le n, u', v', v_i^{'}, u_i^{'}, 1 \le i \le n\}$ be the vertices and $\{a_i, a_i^{'}, 1 \le i \le 2n, \; b_i, b_i^{'}, 1 \le i \le 2n, e_1, e_2, e_3, e_4\}$ be the edges which are denoted as in Fig. 3.9.

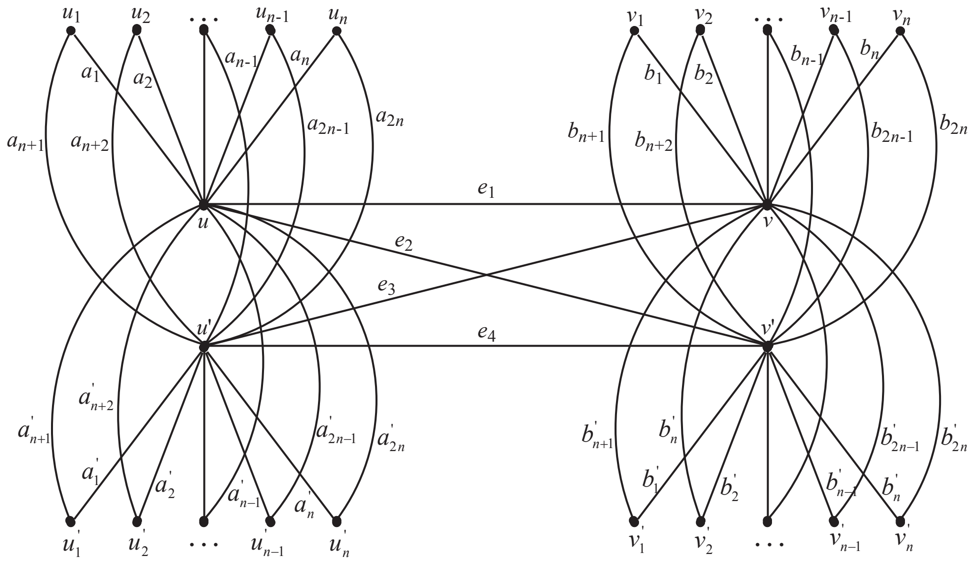


Fig. 3.9: $D_2(B_{n,n})$ with ordinary labeling

First we label the vertices as follows:

Define $f: V \to \{k, k-1, k+1, \ldots, k+2q-1\}$ by $f(u) = k-1$

For $1 \le i \le n$, $f(u_i) = k + 2i - 1$; $f(v) = k + 8n$

For $1 \le i \le n-1$, $f(v_i) = k + 4i$; $f(v_n) = k + 4n + 4$; $f(u') = k + 4n - 1$

For $1 \le i \le n-1$, $f(u_i') = k + 2(n+i) - 1$; $f(u_n') = k + 4n - 2$; $f(v') = k + 8n + 2$

For $1 \le i \le n$, $f(v_i') = k + 8n + 7 - 4i$

Then the induced edge labels are:

For $1 \le i \le n$, $f^+(a_i) = 2k + 2(i-1)$

For $n+1 \le i \le 2n$, $f^+(a_i) = 2k + 2n + 2(i-1)$

For $1 \le i \le n$, $f^+(a_i') = 2k + 6n + 2(i-1)$

For $n+1 \le i \le 2n$, $f^+(a_i') = 2k + 2(i-1)$

For $1 \le i \le n-1$, $f^+(b_i) = 2k + 8n + 4i$; $f^+(b_n) = 2k + 12n + 4$

For $n+1 \le i \le 2n-1$, $f^+(b_i) = 2k + 2 + 4(n+i)$; $f^+(b_{2n}) = 2k + 12n + 6$

For $1 \le i \le n$, $f^+(b_i') = 2k + 16n + 10 - 4i$

For $n+1 \le i \le 2n$, $f^+(b_i') = 2k + 20n + 8 - 4i$

Therefore, $f^+(E) = \{2k, 2k+2, \ldots, 2k+2q-2\}$. So, f is a k-even sequential harmonious labeling and hence, $D_2(B_{n,n})$ is a k-even sequential harmonious graph for any k.

Illustration 3.12: 2-ESHL of $D_2(B_{4,4})$ is shown in Fig. 3.10.

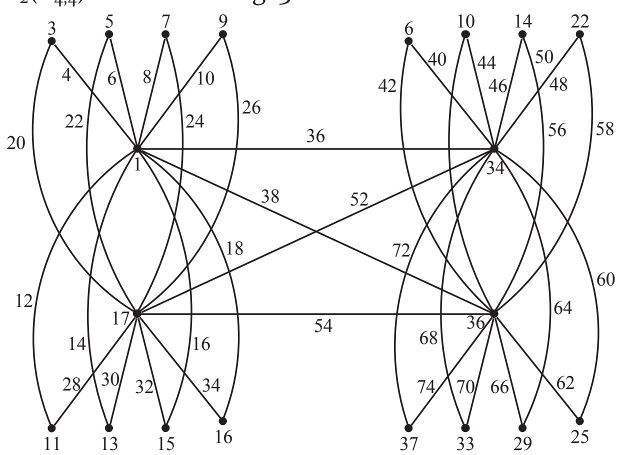


Fig. 3.10: 2-*ESHL* of $D_2(B_{4,4})$

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