ANTIPODAL DOMINATION IN TERMS OF THE ORDER OF GRAPHS

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Abstract : A dominating set $S \subseteq V$ is said to be an Antipodal Dominating Set(ADS) of a connected graph G if there exist vertices $x, y \in S$ such that $d(x, y) = diam(G)$. The minimum cardinality of an ADS is called the Antipodal Domination Number(ADN), and is denoted by γap(G). In this paper, we study the antipodal domination of trees. We also define ADS for disconnected graphs and characterize the graphs G of order n for which γap(G) = n, n-1, n-2.

Keywords: Antipodal Domination, Diameter.

Introduction: Let G = (V, E) be a graph with vertex set V and edge set E. A set $D \subseteq V$ is a dominating set of G if every vertex not in D is adjacent to a vertex in D. The domination number of G, denoted by γ(G), is the minimum cardinality of a dominating set.

A thorough study of domination, with its many variations, appears in [1,2]. We introduced a new domination parameter called Antipodal domination by imposing the antipodal condition on the dominating set[3].Let *G* be a connected graph. A dominating set *S* of *G* is said to be an **Antipodal Dominating Set (ADS)** if there exist vertices $x,y \in S$ such that $d(x,y) = diam(G)$. The minimum cardinality of an ADS is called the **Antipodal Domination Number(ADN),** and is denoted by $\gamma_{ap}(G)$.

It is easy to note that ADS is superhereditary and $\gamma \le \gamma_{ap} \le \gamma + 2$. We have determined γ_{ap} for paths, complete bipartite graphs, generalized wheels, double stars, wounded spiders and Jahangir graphs in [3].

Known results: Result 1[1] For a graph G with no isolated vertices and of order *n*, $\gamma(G) \le n/2$.

Result 2[1] For a graph *G* with no isolated vertices and of order *n*, $\gamma(G) = n/2$ iff the components of *G* are the cycle C_4 or the corona $H \circ K_1$ for any connected graph *H*.

Result 3[3] $\gamma_{ap}(K_{m,n}) = \begin{cases} 2 \text{ if } 2 \in \{m,n\} \text{ or } m = n = 1 \\ 3 \text{ otherwise.} \end{cases}$

Result 4[3] For the Double star $D_{r,s}$,

$$\gamma_{ap}(D_{r,s}) = \begin{cases} 2 \text{ if } r = s = 1 \\ 3 \text{ if } r \ne s \text{ and } \min\{r,s\} = 1 \\ 4 \text{ otherwise.} \end{cases}$$

Antipodal Domination:

Theorem 1 *For any non-trivial connected graph G of order n,* $\gamma_{ap}(G) \le n/2 + 1$.

Proof. Let *S* be a $\gamma-$set of *G*. By Result 1,

$\gamma(G) \le n/2$.

Let *u, v* be vertices of *G* with $d(u, v) = diam(G)$. Then $S \cup \{u, v\}$ is an ADS of *G*.

Case 1 $\gamma(G) < n/2$

Then $\gamma_{ap}(G) \le |S| + 2 < n/2 + 2$.

Case 2 $\gamma(G) = n/2$

By Result 2, G = C_4 or $H \circ K_1$ for any connected graph *H*. Clearly, in both the cases, $\gamma_{ap}(G) = n/2$.

Next, we define ADS for disconnected graphs.

Definition Let *G* be a disconnected graph. Let $G_1, G_2, \ldots, G_k$ be its components. A set *S* is said to be an ADS of *G* if *S* can be written as $S = \bigcup_{i=1}^{k} S_i$, where each S_i is an ADS of G_i. Now $\gamma_{ap}(G) = \sum_{i=1}^{k} \gamma_{ap}(G_i)$.

Theorem 2 *For any graph G* ($\ne 2K_1$) *with* $diam(G) \ge 4$, $\gamma_{ap}(G) \ge 3$.

Proof. Let $diam(G) = k \ge 4$ and S be any ADS.

Case 1 *k* is finite

Let *u,v* be two vertices in *S* such that $d(u,v) = diam(G)$.

Let $P : u = u_0u_1u_2 \ldots u_k = v$ be a diametrical path in *G*.

Now to dominate u_2, we need at least one more vertex in *S*; and so $\gamma_{ap}(G) \ge 3$.

Case 2 $k = \infty$

Now *G* is disconnected.

For every component G_i, $\gamma_{ap}(G_i) = 1$ if $G_i = K_1$; and $\gamma_{ap}(G_i) \ge 2$ otherwise. Since $G \ne 2K_1$, it follows that $\gamma_{ap}(G) \ge 3$.

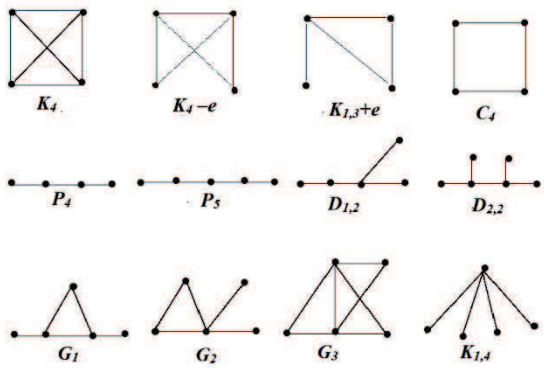
Antipodal domination number of trees:

Theorem 3 *For a tree T,* $\gamma_{ap}(T) = 2$ *iff T is* K_2 *or* $K_{1,2}$ or P_4 (= $D_{1,1}$).
Proof. Using Results 3,4 and Theorem 2, we get the result.
Theorem 4 *Let T be a non-trivial tree with r pendant vertices. Then* $\gamma_{ap}(T) \le n - r + 2$.
Proof. Let u and v be pendant vertices with $d(u,v) = diam(T)$.
Let R be the set of all pendant vertices of T and let $S = (V - R) \cup \{u,v\}$..
Clearly S is an ADS, and so $\gamma_{ap}(T) \le n - r + 2$.
Theorem 5 *Let T be a tree of order* $n \ge 3$, *with r pendant vertices. Then* $\gamma_{ap}(T) = n - r + 2$ iff
(i) Every vertex of T is either a pendant vertex or a support vertex and
(ii) For every pair of vertices x and y with $d(x,y) = diam(T)$, $d(z) \ge 3$ for every $z \in N(x) \cup N(y)$.
Proof. Let $\gamma_{ap}(T) = n - r + 2$.
Let R denote the set of all pendant vertices in T.
(i) Assume the contrary that (i) does not hold. Let w be a vertex that is neither a pendant nor a support.
Then w is adjacent to a non-pendant vertex.
Let u and v be pendant vertices with $d(u,v) = diam(T)$.
Then $(V - R) \cup \{u,v\} - \{w\}$ is an ADS; and so $\gamma_{ap}(T) \le n - r + 1$, a contradiction.
(ii) Assume the contrary that (ii) does not hold.
Then there exists a pair of vertices u and v with $d(u, v) = diam(T)$, such that $d(w) \le 2$ for some vertex $w \in N(u) \cup N(v)$.
Since $d(u, v) = diam(T)$ and T is a tree, it follows that $d(w) = 2$ and w lies in the (u,v)-path.
Now $(V - R) \cup \{u,v\} - \{w\}$ is an ADS; and $\gamma_{ap}(T) \le n - r + 1$, a contradiction.
Converse is obvious.

Some Characterizations:

Theorem 6 *Let G be a connected graph of order n. Then (i)* $\gamma_{ap}(G) = n$ *iff G is* K_2 *or* K_1.
(ii) $\gamma_{ap}(G) = n-1$ *iff G is* K_3, P_3 *or* $K_{1,3}$.
Proof. (i) Let $\gamma_{ap}(G) = n$.
Now using Theorem 1, $n \le n/2 + 1$.
Hence $G = K_2$ *or* K_1 .
Converse is obvious.
(ii)Let $\gamma_{ap}(G) = n-1$.
Now $n - 1 \le n/2 + 1$ implies that $n \le 4$.
Using (i), $n = 3$ *or* 4.
When $n = 3$, $G = P_3$ *or* K_3.
When $n = 4$, diam(G) ≤ 3.
Since $\gamma_{ap}(K_n) = 2$, $G \ne K_n$.
Hence $diam(G) = 2$ or 3.
When $diam(G) = 3$, let $u_0u_1u_2u_3$ be a diametrical path. Then $V - \{u_1, u_2\}$ is an ADS, a contradiction. Hence $diam(G) = 2$.
Now we can easily check that $G = K_{1,3}$.
Converse is obvious.

Corollary 7 *Let G be a graph of order n. Then*
(i) $\gamma_{ap}(G) = n$ *iff* $G = rK_1 \cup sK_2$, *where* $r + 2s = n$, *and* $r, s \ge 0$.
(ii) $\gamma_{ap}(G) = n-1$ *iff G is* $K_3 \cup r_1K_1 \cup s_1K_2$ *or*
$P_3 \cup r_1K_1 \cup s_1K_2$ *or* $K_{1,3} \cup r_2K_1 \cup s_2K_2$, where $r_1 + 2s_1 = n-3$, $r_2 + 2s_2 = n - 4$ and
$r_1, r_2, s_1, s_2 \ge 0$.
Proof. (i) follows obviously from Theorem 6.
(ii) Let $\gamma_{ap}(G) = n-1$.
First note that $n \ge 2$. Let $G_1, G_2, \ldots, G_l$ be the components of G of order $n_1, n_2, \ldots, n_l$ respectively.
Since $\gamma_{ap}(G) = n-1$, $\gamma_{ap}(G_i) = n_i-1$, for exactly one i, $1 \le i \le l$, and $\gamma_{ap}(G_j) = n_j$ for all $j \ne i$, $1 \le j \le l$; and the result follows from Theorem 6.
Theorem 8 *Let* $G = (V, E)$ *be a connected graph of order n. Then* $\gamma_{ap}(G) = n-2$ *iff* $G \in A$, *where A is the collection of graphs in Fig 1 .*
Proof. Let $\gamma_{ap}(G) = n-2$.



Graphs in Collection $\mathscr{A}$

Fig. 1

Now $n\text{-}2 \le n/2 + 1$ implies that $n \le 6$.
Using Theorem6, $n \ge 4$. Also note that all graphs of order 4, other than $K_{1,3}$, have $\gamma_{ap}(G) = 2$, and they belong to A .
Next, when $n=5$ or 6, $diam(G) = k \le 5$.
Let $u_0 u_1 u_2 \ldots u_k$ be a diametrical path in G.
Case 1 $k=5$
Then $V - \{u_1, u_2, u_4\}$ is an ADS, a contradiction.
Case 2 $k=4$
Then $\mathbf{G = P_5} \in A$ or there exists a vertex $v \in V - \{u_0, u_1, u_2, u_3, u_4\}$, that is adjacent to u_i, for some i, $0 \le i \le 4$.
If vu_0 or $vu_4 \in E$, then $V - \{v, u_1, u_2\}$ is an ADS, a contradiction.
If $vu_i \in E$, for $1 \le i \le 3$, then $V\text{-}\{v, u_1, u_2, u_3\} \cup \{u_i\}$ is an ADS, a contradiction. **Case 3** $k=3$
Let $v \in V - \{u_0, u_1, u_2, u_3\}$ be adjacent with u_i, for some i, $0 \le i \le 3$..
If vu_0 or $vu_3 \in E$, then $V - \{v, u_1, u_2\}$ is an ADS, a contradiction.
Hence every $v \in V - \{u_0, u_1, u_2, u_3\}$ is not adjacent with u_0 and u_3; and so $\boldsymbol{d(u_0) = 1}$ and $\boldsymbol{d(u_3) = 1}$.
Now we have two cases: (i) $vu_1 \in E$ and $vu_2 \notin E$ (ii) $vu_1, vu_2 \in E$ (The case $vu_2 \in E$ and $vu_1 \notin E$ is similar to case (i)).
Case 3.1 $vu_1 \in E$ and $vu_2 \notin E$
Then $\boldsymbol{G = D_{1,2}} \in A$ or there exists a vertex $w \in V - \{u_0, u_1, u_2, u_3, v\}$ such that wu_1, wu_2 or $wv \in E$.
Case 3.1.1 $wu_1 \in E$
Now $V - \{v, w, u_2\}$ is an ADS, a contradiction.
Case 3.1.2 $wv \in E$
Then $V - \{u_1, u_2, w\}$ is an ADS, a contradiction.
Case 3.1.3 $wu_2 \in E$
Then $\boldsymbol{G = D_{2,2}} \in A$.
Case 3.2 $vu_1, vu_2 \in E$
Then $\boldsymbol{G = G_1} \in A$ or there exists a vertex $w \in V - \{u_0, u_1, u_2, u_3, v\}$ such that wu_1, wu_2 or $wv \in E$.
If $wu_1 \in E$ (or $wu_2 \in E$), then $V - \{v, w, u_2\}$ (or $V - \{v, w, u_1\}$) is an ADS, a contradiction.
If $wv \in E$, then $V - \{w, u_1, u_2\}$ is an ADS, a contradiction.
Case 4 $k = 2$
If there exist two vertices x, y in $V - \{u_0, u_1, u_2\}$ adjacent with u_0 (or u_2), then $V - \{x, y, u_1\}$ is an ADS, a contradiction.
Hence $\boldsymbol{d(u_0) \le 2}$ and $\boldsymbol{d(u_2) \le 2}$.
Moreover, if $d(u_1) \ge 5$, let $N(u_1) \supseteq \{u_0, u_2, z_1, z_2, z_3\}$. Then $V - \{z_1, z_2, z_3\}$ is an ADS, a contradiction. Hence $\boldsymbol{d(u_1) \le 4}$.
Let $v \in V - \{u_0, u_1, u_2\}$ be adjacent with u_i, for some i, $i = 0,1,2$.
Case 4.1 $vu_0 \in E$ and $vu_1, vu_2 \notin E$
Then $d(v, u_1) = 2$ and $d(v, u_2) = 2$.
Let $w \in V - \{u_0, u_1, u_2, v\}$ be adjacent with u_1, u_2 or v. Note that $wu_0 \notin E$.
If $wu_1 \in E$ (or $wu_2 \in E$) , then $V - \{u_0, u_2, w\}$ (or $V - \{u_0, u_1, w\}$) is an ADS; if $wv \in E$, then $V - \{u_0, u_1, w\}$ is an ADS, a contradiction.
Case 4.2 $vu_2 \in E$ and $vu_0, vu_1 \notin E$

This case is similar to case 4.1.

Case 4.3 $vu_0, vu_2 \in E$ and $vu_1 \notin E$

There exists a vertex $w \in V - \{u_0, u_1, u_2, v\}$ such that w is adjacent with u_1 or v.Then $V - \{u_0, u_2, w\}$ is an ADS, a contradiction.

Case 4.4 $vu_1, vu_2 \in E$ and $vu_0 \notin E$

There exists $w \in V - \{u_0, u_1, u_2, v\}$ such that w is adjacent with v, u_1 or u_0.

Case 4.4.1 $wv \in E$ *a*

Then $V - \{u_1, u_2, w\}$ is an ADS, a contradiction.

Case 4.4.2 $wu_1 \in E$

If $wu_0 \in E(G)$, then $V - \{u_1, v, w\}$ is an ADS, a contradiction.

So $\boldsymbol{G = G_2} \in A$ or there exists $z \in V\text{-}\{u_0, u_1, u_2, v, w\}$. Since $d(u_1) \leq 4$, z is not adjacent with u_1. Now $d(z,u_1) = 2$ and $V - \{u_0, u_2, v\}$ is an ADS, a contradiction.

Case 4.4.3 $wu_0 \in E$

Since $wu_2 \notin E$, using case 4.1, $wu_1 \in E$, and this case is already dealt in case 4.4.2.

Case 4.5 $vu_0, vu_1 \in E$ and $vu_2 \notin E$

Similar to case 4.4.

Case 4.6 $vu_0, vu_1, vu_2 \in E$

There exists $w \in V - \{u_0, u_1, u_2, v\}$ such that w is adjacent with u_1 or v.

Case 4.6.1 $wv \in E$ and $wu_1 \notin E$

Now $d(w,u_1) = 2$ and $V - \{v, u_0, u_2\}$ is an ADS, a contradiction.

Case 4.6.2 $wv \notin E$ and $wu_1 \in E$

Now $d(v,w) = 2$ and $V - \{u_0, u_1, u_2\}$ is an ADS, a contradiction.

Case 4.6.3 $wv, wu_1 \in E$

Then $\boldsymbol{G = G_3} \in A$ or there exists $x \in V - \{u_0, u_1, u_2, v, w\}$ such that x is adjacent with v or w.

Then $V - \{u_0, u_1, x\}$ is an ADS, a contradiction.

Case 4.7 $vu_1 \in E$ and $vu_0, vu_2 \notin E$

Then $d(v,u_2) = 2$. Let $w \in V - \{u_0, u_1, u_2, v\}$.

If $wu_1 \notin E$, then $V - \{u_0, u_2, v\}$ is an ADS, a contradiction. Hence $\boldsymbol{wu_1 \in E}$ for all $w \in V - \{u_0, u_1, u_2, v\}$.

Since $d(u_1) \leq 4$, it follows that $\boldsymbol{n = 5}$. The adjacency between u_0, u_2 and w leads to cases similar to cases 4.4, 4.5 and 4.6. Hence we can assume that $wu_0, wu_2 \notin E$. When $vw \in E$ or $vw \notin E$, $\boldsymbol{G = G_3}$ *or* $\boldsymbol{K_{1,4}}$ and $\boldsymbol{G} \in A$.

Case 5 $\boldsymbol{k = 1}$

Then G is complete, contradicting $n \geq 5$.

Converse is obvious.

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