ON A CONTIGUOUS EXTENSION OF A CLASS OF SUMMATION FORMULAS INVOLVING THE GENERALIZED 2F2 HYPERGEOMETRIC POLYNOMIAL

YASHOVERDHAN VYAS, KALPANA FATAWAT

Abstract: Extentions of Kummer's first transformation, Gauss and Kummer summation, and its contigous results are applied to obtain a contiguous extension of a class of summation formulas involving the generalized ${}_2F_2$ hypergeometric polynomial, which have not previously appeared in literature. Certain well-known results and other interesting special cases involving Laguerre polynomial, Bessel function and hypergeometric functions are also obtained from our main findings.

Keywords: generalized hypergeometric series; extension of Gauss and Kummer summation theorems; polynomial

MSC: 33C15; 33C20

Introduction: The generalized hypergeometric function with p numeratorial and q denominatorial parameters is defined by the series

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots\ldots a_p\ ; \\ b_1, b_2, \ldots. b_q\ ; \end{matrix}\ x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots\ldots (a_p)_n}{(b_1)_n (b_2)_n \ldots\ldots (b_q)_n} \frac{x^n}{n!} \tag{1.1}$$

when $q = p$, this series converges for $|x| < 0$, but when q = p − 1 convergence occurs when |x| < 1(unless the series terminates). The pochhammer symbol used above is defined for any complex number α ϵ C by

$$(\alpha)_n = \begin{pmatrix} \alpha(\alpha+1)\ldots.(\alpha+n-1) & n\ \epsilon\ N \\ 1 & n = 0 \end{pmatrix} \tag{1.2}$$

Also, $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ (1.3)

It is observed that the results obtained by reducing a generalized hypergeometric function to gamma function are very important.

Some of the classical summation theorems, their generalizations, and their extensions are here to mention so that the paper can be self-contained:

Guass summation theorem [2,7]:

$$ {}_2F_1\left[\begin{matrix} a,\ b\ ; \\ c\ ; \end{matrix}\ 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{1.4}$$

Provided that R(c-a-b) > 0

Kummer's summation theorem[2,7]:

$$ {}_2F_1\left[\begin{matrix} a,\ b\ ; \\ 1+a-b\ ; \end{matrix}\ -1\right] = \frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)} \tag{1.5}$$

Contiguous Kummer's summation theorems [7]:

$$ {}_2F_1\left[\begin{matrix} a, & b\ ; \\ & a-b\ ; \end{matrix}\ -1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(a-b)}{2^a}\left(\frac{1}{\Gamma\left(\frac{a}{2}\right)\Gamma(\frac{a}{2}-b+\frac{1}{2})} + \frac{1}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma(\frac{a}{2}-b)}\right) \tag{1.6}$$

$$ {}_2F_1\left[\begin{matrix} a, & b\ ; \\ a-b-2\ ; \end{matrix}\ -1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(a-b-2)}{2^a}\left(\frac{2a-3b-4}{\Gamma\left(\frac{a}{2}\right)\Gamma(\frac{a}{2}-b-\frac{1}{2})} + \frac{(2a-b-2)}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma(\frac{a}{2}-b-1)}\right) \tag{1.7}$$

Extension of kummer's summation theorem [4]:

$$ {}_3F_2\left[\begin{matrix} a, & b, d+1\ ; \\ & 2+a-b, d\ ; \end{matrix}\ -1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(2+a-b)}{2^a(1-b)}\left(\frac{(\frac{1+a-b}{d}-1)}{\Gamma\left(\frac{a}{2}\right)\Gamma(\frac{a}{2}-b+\frac{3}{2})} + \frac{(1-\frac{a}{d})}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma(1+\frac{a}{2}-b)}\right) \tag{1.8}$$

Extension of contiguous Kummer's theorem(1.7):

$$ {}_3F_2\left[\begin{matrix} a, & b, d+1\ ; \\ & a-b, d\ ; \end{matrix}\ -1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(a-b)}{2^a}\left(\frac{(1-\frac{b}{d})}{\Gamma\left(\frac{a}{2}\right)\Gamma(\frac{a}{2}-b+\frac{1}{2})} + \frac{1}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma(\frac{a}{2}-b)}\right) \tag{1.9}$$

Generalized Kummer's summation theorem [9]:

$$ {}_2F_1\left[\begin{matrix} a, & b\ ; \\ 1+a-b+j\ ; \end{matrix}\ -1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1-b)\Gamma(1+a-b+j)}{2^a\Gamma(1-b+\epsilon_j)}\left(\frac{A_j(a,b)}{\Gamma\left(\frac{a}{2}+\delta_{j+1}\right)\Gamma\left(\frac{a}{2}-b+\frac{1}{2}j+1\right)} + \frac{B_j(a,b)}{\Gamma\left(\frac{a}{2}+\delta_j\right)\Gamma\left(\frac{a}{2}-b+\frac{1}{2}j+\frac{1}{2}\right)}\right) \tag{1.10}$$

Where,

$\epsilon_j \equiv \frac{1}{2}(j+|j|)$, $\delta_j \equiv \frac{1}{2}j - \left[\frac{1}{2}j\right]$

Extension of contiguous Kummer's summation theorem (1.7):

$$ {}_3F_2\left[\begin{matrix} a, & b, d+2\ ; \\ & a-b, d\ ; \end{matrix}\ -1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(a-b)}{2^a}\left[\frac{\{1-\frac{2b}{d}+\frac{b(a+1)}{2d(d+1)}\}}{\Gamma\left(\frac{a}{2}\right)\Gamma(\frac{a}{2}-b+\frac{1}{2})} + \frac{\{1-\frac{ab}{2d(d+1)}\}}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma(\frac{a}{2}-b)}\right] \tag{1.11}$$

Extension of Kummer's summation theorem:

$$ {}_3F_2\left[\begin{matrix} a, & b, d+2\ ; \\ 3+a-b, & d\ ; \end{matrix}\ -1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(3+a-b)}{2^a(1-b)(2-b)}\left[\frac{\left\{-2+\frac{2(2a-b+2)}{d}-\frac{2(a-b+2)(a+1)}{d(d+1)}\right\}}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a}{2}-b+\frac{3}{2}\right)} + \frac{\{(a-b+1)+\frac{a(-2a+3b-4)}{d}+\frac{a\{2(a-b+3)(a-b+1)-(b+1)(b-2)\}}{2d(d+1)}\}}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-b+2\right)}\right] \tag{1.12}$$

Proof: in order to arrive (1.11) we proceed as follows. Denoting the left-hand side of (1.11) by S, expressing ${}_3F_2$ as a series, after some simplification, we get

$$S = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n\,(-1)^n}{(a-b)_n n!}\left(1+\frac{n}{d}\right)\left(1+\frac{n}{d+1}\right)$$

$$S = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n\,(-1)^n}{(a-b)_n\, n!} + \frac{2}{d}\sum_{n=1}^{\infty} \frac{(a)_n (b)_n (-1)^n}{(a-b)_n\,(n-1)!} + \frac{1}{d(d+1)}\sum_{n=2}^{\infty} \frac{(a)_n (b)_n (-1)^n}{(a-b)_n\,(n-2)!}$$

$$S=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n\,(-1)^n}{(a-b)_n\,n!}-\frac{2ab}{d(a-b)}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n(-1)^n}{(a-b+1)_n\,(n)!}+\frac{ab(a+1)(b+1)}{d(d+1)(a-b)(a-b+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n(-1)^n}{(a-b+2)_n\,(n)!}$$

It can be observe that the first term of above equation can be evaluated with the help of (1.6) while middle term can be evaluated with the help of (1.5) by replacing a with a+1 and b with b+1 and third term can be evaluated with the help of (1.8) by replacing a with a+2, b with b+2 and d→ ∞. After some simplification right hand side of (1.11) can be obtained.

If d = $a-b-2$, we recover the result (1.7) that can be obtained by taking j = -3 in (1.10). The result (1.10) is given by Kim et. al.[9]

To prove (1.12) consider left-hand side of (1.12), i.e.

$$ {}_3F_2\left[\begin{matrix}a, & b, d+2\,; \\ 3+a-b, & d\,;\end{matrix}\; -1\right]$$

$$=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n\,(-1)^n}{(3+a-b)_n n!}\left(1+\frac{n}{d}\right)\left(1+\frac{n}{d+1}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n\,(-1)^n}{(3+a-b)_n\,n!}+\frac{2}{d}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n(-1)^n}{(3+a-b)_n\,(n-1)!}+\frac{1}{d(d+1)}\sum_{n=2}^{\infty}\frac{(a)_n(b)_n(-1)^n}{(3+a-b)_n\,(n-2)!}$$

$$=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n\,(-1)^n}{(3+a-b)_n\,n!}-\frac{2ab}{d(3+a-b)}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n(-1)^n}{(4+a-b)_n\,(n)!}+\frac{ab(a+1)(b+1)}{d(d+1)(3+a-b)(4+a-b)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n(-1)^n}{(5+a-b)_n\,(n)!}$$

It can be observed that the first term of above equation can be evaluated with the help of (1.10) by taking $j=2$. The second term can be solved with the aid of (1.10) by taking $a=a+1$, b = b + 1 and $j=3$. Similarly third term can also be calculated with the help of (1.10) by taking $a=a+2$, b = b + 2 and taking $j=4$. The value of respective $A_j(a,b) and\ B_j(a,b)$ can be obtained from the table.[Table 1, 9]

After this, we get

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(3+a-b)\Gamma(1-b)}{2^a\Gamma(3-b)}\left(\frac{a-b+1}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-b+2\right)}-\frac{2}{\Gamma\left(\frac{a}{2}\right)\,\Gamma\left(\frac{a}{2}-b+\frac{3}{2}\right)}\right)$$

$$-\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(4+a-b)\Gamma(-b)ab}{2^a\Gamma(3-b)d(3+a-b)}\left(\frac{-2a+3b-4}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-b+2\right)}+\frac{2a-b+2}{\Gamma\left(\frac{a}{2}+1\right)\,\Gamma\left(\frac{a}{2}-b+\frac{3}{2}\right)}\right)$$

$$+\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(5+a-b)\Gamma(-1-b)}{2^a\Gamma(3-b)(3+a-b)(4+a-b)}\frac{a(a+1)b(b+1)}{4d(d+1)}\left(\frac{2(a-b+3)(a-b+1)-(b+1)(b-2)}{\Gamma\left(\frac{a}{2}+\frac{3}{2}\right)\Gamma(\frac{a}{2}-b+2)}-\frac{4(a-b+2)}{\Gamma\left(\frac{a}{2}+1\right)\,\Gamma(\frac{a}{2}-b+\frac{3}{2})}\right)$$

After some simplification, desired result is obtained.

It is important to mention here about Laguerre polynomial which is extensively used in many branches of pure and applied mathematics, engineering and mathematical physics. It is a terminating form of confluent hypergeometric function ${}_1F_1$ defined by

$$L_n^{(v)}(x)=\frac{(v+1)_n}{n!}\,{}_1F_1\left[\begin{matrix}-n; \\ v+1;\end{matrix}\; x\right], \quad (1.13)$$

On the other hand, by elementary manipulation of series, Kim et al. [9] have obtained the following general transformation involving the generalized hypergeometric function viz.:

$$\sum_{n=0}^{\infty}\frac{(a_1)_n\dots\dots\dots(a_p)_n}{(b_1)_n\dots\dots\dots(b_q)_n}\frac{(d)_n}{n!}x^n y^n\,{}_1F_1\left[\begin{matrix}d+n\,; \\ f;\end{matrix}\; x\right]$$

$$=\sum_{n=0}^{\infty}\frac{(d)_n}{(f)_n}\frac{x^n}{n!}\,{}_{p+2}F_q\left[\begin{matrix}-n,1-f-n,a_1\dots\dots a_p\,; \\ b_1\dots\dots b_q;\end{matrix}\; y\right] \quad (1.14)$$

Using Kummer's first transformation [2], viz.

$${}_1F_1\left[\begin{matrix}a\,; \\ c\,;\end{matrix}\; x\right]=e^x\,{}_1F_1\left[\begin{matrix}c-a\,; \\ c\,;\end{matrix}\; -x\right] \quad (1.15)$$

in (1.16) and employing Kummer's summation theorem (1.5) and Gauss summation theorem (1.4), they have also obtained the following interesting result summation formula involving the Laguerre polynomial viz.:

$$e^{-x}\sum_{n=0}^{\infty}L_n^{(v)}(x)\frac{x^n}{(v+1)_n}={}_0F_1\left[\begin{matrix}-\,; \\ v+1\,;\end{matrix}\; -x^2\right] \quad (1.16)$$

where v ≠ -1,-2....

The following results established by paris[11] will be required for our new results

$${}_2F_2\left[\begin{matrix}a, & c+1\,; \\ b, & c\,;\end{matrix}\; x\right]=e^x\,{}_2F_2\left[\begin{matrix}b-a-1, & f+1\,; \\ b, & f;\end{matrix}\; -x\right] \quad (1.17)$$

with f = $\frac{c(1+a-b)}{a-c}$ which is extension of Kummer's first transformation (1.17)

also,

$${}_2F_2\left[\begin{matrix}a, & c+1\,; \\ b, & c\,;\end{matrix}\; x\right]={}_1F_1\left[\begin{matrix}a\,; \\ b\,;\end{matrix}\; x\right]+\frac{ax}{bc}\,{}_1F_1\left[\begin{matrix}a+1\,; \\ b+1\,;\end{matrix}\; x\right] \quad (1.18)$$

In which $_2F_2$ is expressed as a sum of two $_1F_1$ functions [4, p. 585(1)]

An important formula that can be derived with the help of (1.18) is given by

$$ {}_2F_2\left[\begin{matrix} a, & d+2\,; \\ & b,\ d\,; \end{matrix}\ x\right] = {}_2F_2\left[\begin{matrix} a, & d+1\,; \\ & b,\ d\,; \end{matrix}\ x\right] + \frac{ax}{bd}\,{}_2F_2\left[\begin{matrix} a+1, d+2\,; \\ b+1, \quad d+1\,; \end{matrix}\ x\right] $$

Now using (1.17), (1.19) can be written as

$$ {}_2F_2\left[\begin{matrix} a, & d+2\,; \\ & b,\ d\,; \end{matrix}\ x\right] = e^x\,{}_2F_2\left[\begin{matrix} b-a-1, & f+1\,; \\ b, & f\,; \end{matrix}\ -x\right] + \frac{ax}{bd}e^x\,{}_2F_2\left[\begin{matrix} b-a-1, & f'+1\,; \\ b+1, & f'\,; \end{matrix}\ -x\right] $$

Where $f = \frac{d(1+a-b)}{a-d}$ and $f' = \frac{(d+1)(1+a-b)}{a-d}$

Replacing b with $b+1$

$$ {}_2F_2\left[\begin{matrix} a, & d+2\,; \\ & b+1,\ d\,; \end{matrix}\ x\right] = e^x\,{}_2F_2\left[\begin{matrix} b-a, & f+1\,; \\ b+1, & f\,; \end{matrix}\ -x\right] + \frac{ax}{(b+1)d}e^x\,{}_2F_2\left[\begin{matrix} b-a, & f'+1\,; \\ b+2, & f'\,; \end{matrix}\ -x\right] \quad (1.19) $$

Where $f = \frac{d(a-b)}{a-d}$ and $f' = \frac{(d+1)(a-b)}{a-d}$

by establishing a general transformation formula, generalization of (1.12) is performed which is used in deriving new summation formula for the generalized hypergeometric $_2F_2$ polynomial by employing extension of Gauss summation theorem, Kummer's summation theorem, it's extension and contiguous results.

Main transformation formula: The transformation formula involving the generalized hypergeometric function to be established is

$$ \sum_{n=0}^{\infty}\frac{(a_1)_n\ldots\ldots\ldots(a_p)_n}{(b_1)_n\ldots\ldots\ldots(b_q)_n}\frac{(d)_n}{n!}x^n y^n\,{}_2F_2\left[\begin{matrix} d+n, e+2\ ; \\ f+1, e; \end{matrix}\ x\right] = \sum_{n=0}^{\infty}\frac{(d)_n\,(e+2)_n}{(f+1)_n\,(e)_n}\frac{x^n}{n!}\,{}_{p+3}F_{q+1}\left[\begin{matrix} -n, 1-e-n, -f-n, a_1\ldots\ldots a_p\ ; \\ -1-e-n, b_1\ldots\ldots b_q; \end{matrix}\ y\right] \quad (2.1) $$

Proof: Denoting left-hand side of (2.1) by S and expressing $_2F_2$ as a series, gives

$S = \sum_{n=0}^{\infty}\frac{(a_1)_n\ldots\ldots(a_p)_n}{(b_1)_n\ldots\ldots(b_q)_n}\frac{(d)_n}{n!}x^n y^n\sum_{n=0}^{\infty}\frac{(d+n)_m\,(e+2)_m x^m}{(f+1)_m\ (e)_m\ m!}$

Using $(d)_n(d+n)_m = (d)_{m+n}$

Equation becomes

$S = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots\ldots(a_p)_n}{(b_1)_n\ldots\ldots(b_q)_n}\frac{(d)_{m+n}(e+2)_m}{(f+1)_m(e)_m n!m!}x^{n+m}y^n$

Now replacing m by $m-n$ and making use of double series manipulation

$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k)$

Equation becomes

$S =$
$\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(a_1)_n\ldots\ldots(a_p)_n}{(b_1)_n\ldots\ldots(b_q)_n}\frac{(d)_m(e+2)_{m-n}}{(f+1)_{m-n}(e)_{m-n}n!m-n!}x^m y^n$

Using identities

$(a)_{m-n} = \frac{(-1)^n(a)_m}{(1-a-m)_n}$

$(m-n)! = \frac{(-1)^n\, m!}{(-m)_n}$

After simplification, following summation is obtained.

$$ S = \sum_{m=0}^{\infty}\frac{(d)_m\,(e+2)_m x^m}{(f+1)_m\ (e)_m\ m!}\sum_{n=0}^{m}\frac{(a_1)_n\ldots\ldots(a_p)_n}{(b_1)_n\ldots\ldots(b_q)_n}\frac{(1-e-m)_n(-f-m)_n(-m)_n}{(-1-e-m)_n n!}y^n $$

New class of summation formula involving the generalized hypergeometric $_2F_2$ polynomial

Theorem 1 :

$$ e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{n!}\,{}_2F_2\left[\begin{matrix} -n, g+1\ \ ; \\ v+2, g; \end{matrix}\ x\right] + \frac{(v+1+n)(-x)}{e(v+2)}\,{}_2F_2\left[\begin{matrix} -n, g'+1\ \ ; \\ v+3, \quad g'; \end{matrix}\ x\right] $$
$$ = {}_3F_4\left[\begin{matrix} \frac{v}{2}+\frac{1}{2}, \frac{e}{2}+1, \frac{e}{2}+\frac{3}{2}; \\ v+1, \frac{v}{2}+\frac{3}{2}, \frac{e}{2}, \frac{e}{2}+\frac{1}{2}; \end{matrix}\ -x^2\right] + \frac{x^2}{e(e+1)}\,{}_0F_1\left[\begin{matrix} -\,; \\ v+2; \end{matrix}\ -x^2\right] $$
$$ + \frac{(e+2)x}{(v+2)e}\,{}_3F_4\left[\begin{matrix} \frac{v}{2}+1, \frac{e}{2}+2, \frac{e}{2}+\frac{3}{2}; \\ v+2, \frac{v}{2}+2, \frac{e}{2}+1, \frac{e}{2}+\frac{1}{2}; \end{matrix}\ -x^2\right] - \frac{2x}{e}\,{}_1F_2\left[\begin{matrix} \frac{e}{2}+\frac{3}{2}; \\ v+2, \frac{e}{2}+\frac{1}{2}; \end{matrix}\ -x^2\right] - \frac{x^3}{(e+1)e(v+2)}\,{}_0F_1\left[\begin{matrix} -\,; \\ v+3; \end{matrix}\ -x^2\right] $$

Where,

$g = \frac{en}{v+1+n-e}$ and $g' = \frac{(e+1)n}{v+1+n-e}$

Proof: if we apply the extension of kummer's first transformation (1.19) to left hand side of (2.1) we obtain

$$ e^x\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots\ldots\ldots(a_p)_n}{(b_1)_n\ldots\ldots\ldots(b_q)_n}\frac{(d)_n}{n!}x^n y^n\,{}_2F_2\left[\begin{matrix} f-d-n, g+1\ \ ; \\ f+1, g; \end{matrix}\ -x\right] + \frac{(d+n)x}{e(f+1)} + \left[\begin{matrix} f-d-n, g'+1\ \ ; \\ f+2, \quad g'; \end{matrix}\ -x\right] $$
$$ = \sum_{n=0}^{\infty}\frac{(d)_n\,(e+2)_n}{(f+1)_n\,(e)_n}\frac{x^n}{n!}\,{}_{p+3}F_{q+1}\left[\begin{matrix} -n, 1-e-n, -f-n, a_1\ldots\ldots a_p\ ; \\ -1-e-n, b_1\ldots\ldots b_q; \end{matrix}\ y\right] \quad (3.1) $$

Where $g = \frac{e(d+n-f)}{d+n-e}$, $g' = \frac{(e+1)(d+n-f)}{d+n-e}$

Put $d = f$ in (3.1)

$$e^{x}\sum_{n=0}^{\infty}\frac{(a_1)_n\dots\dots\dots(a_p)_n}{(b_1)_n\dots\dots\dots(b_q)_n}\frac{(f)_n}{n!}x^n y^n\{\,{}_2F_2\left[\begin{matrix}-n,g+1\;; \\ f+1,g;\end{matrix}\quad -x\right]$$

$$+\frac{(f+n)x}{e(f+1)}\,{}_2F_2\left[\begin{matrix}-n,g'+1\;; \\ f+2,g';\end{matrix}\quad -x\right]\}$$

$$=\sum_{n=0}^{\infty}\frac{(f)_n\,(e+2)_n}{(f+1)_n\,(e)_n}\frac{x^n}{n!}\,{}_{p+3}F_{q+1}\left[\begin{matrix}-n,1-e-n,-f-n,a_1\dots\dots a_p\;; \\ -1-e-n,b_1\dots\dots b_q;\end{matrix}\quad y\right]$$

Replace x with –x and put f = v+1

$$e^{-x}\sum_{n=0}^{\infty}\frac{(a_1)_n\dots\dots\dots(a_p)_n}{(b_1)_n\dots\dots\dots(b_q)_n}\frac{(v+1)_n}{n!}(-x)^n y^n\{\,{}_2F_2\left[\begin{matrix}-n,g+1\;; \\ v+2,\quad g;\end{matrix}\quad x\right]$$

$$+\frac{(v+1+n)(-x)}{e(v+2)}\,{}_2F_2\left[\begin{matrix}-n,g'+1\;; \\ v+3,g';\end{matrix}\quad x\right]\}$$

$$=\sum_{n=0}^{\infty}\frac{(v+1)_n\,(e+2)_n}{(v+2)_n\,(e)_n}\frac{(-x)^n}{n!}\,{}_{p+3}F_{q+1}\left[\begin{matrix}-n,1-e-n,-1-v-n,a_1\dots\dots a_p\;; \\ -1-e-n,b_1\dots\dots b_q;\end{matrix}\quad y\right] \qquad (3.2)$$

Now, set p=0, q=1, b_1 = v+1 and y = -1 in (3.2)

$$e^{-x}\sum_{n=0}^{n=\infty}\frac{x^n}{n!}\left\{2F2\left[\begin{matrix}-n,g+1\;; \\ v+2,\quad g;\end{matrix}\quad x\right]+\frac{(v+1+n)(-x)}{e(v+2)}\,2F2\left[\begin{matrix}-n,g'+1\;; \\ v+3,g';\end{matrix}\quad x\right]\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(v+1)_n\,(e+2)_n}{(v+2)_n\,(e)_n}\frac{(-x)^n}{n!}\,{}_3F_2\left[\begin{matrix}-n,-1-v-n,1-e-n\;; \\ v+1,-1-e-n;\end{matrix}\quad -1\right] \qquad (3.3)$$

${}_3F_2$ on Right hand side of (3.3) can be solved by (1.11)

$$=\sum_{n=0}^{\infty}\frac{(v+1)_n\,(e+2)_n}{(v+2)_n\,(e)_n}\frac{(-x)^n}{n!}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(v+1)}{2^{-n}}\left(\frac{\left(1-\frac{2(v+1+n)}{1+e+n}+\frac{(v+1+n)(n-1)}{2(1+e+n)(e+n)}\right)}{\Gamma\left(\frac{-n}{2}\right)\Gamma\left(\frac{n}{2}+v+\frac{3}{2}\right)}+\frac{\left(1-\frac{n(v+1+n)}{2(1+e+n)(e+n)}\right.}{\Gamma\left(\frac{-n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+v+1\right)}\right) \qquad (3.4)$$

Now, separating the terms appearing on the right hand side of (3.7) into even and odd power of x and making use of the following identities:

$\Gamma(a-n)=\frac{(-1)^n\Gamma(a)}{(1-a)_n}$, $2n! = 2^{2n}n!\left(\frac{1}{2}\right)_n$, $(2n+1)! = 2^{2n}n!\left(\frac{3}{2}\right)_n$, $(a)_{2n}=2^{2n}(\frac{a}{2})_n(\frac{a}{2}+\frac{1}{2})_n\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n$, $(a)_{2n+1}=a2^{2n}\left(\frac{a}{2}+1\right)_n\left(\frac{a}{2}+\frac{1}{2}\right)_n$,

After some straight forward calculation, result is

$$e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{n!}\,{}_2F_2\left[\begin{matrix}-n,g+1\;; \\ v+2,\quad g;\end{matrix}\quad x\right]+\frac{(v+1+n)(-x)}{e(v+2)}\,{}_2F_2\left[\begin{matrix}-n,g'+1\;; \\ v+3,g';\end{matrix}\quad x\right]\}$$

$$=\,{}_3F_4\left[\begin{matrix}\frac{v}{2}+\frac{1}{2},\frac{e}{2}+1,\frac{e}{2}+\frac{3}{2}; \\ v+1,\frac{v}{2}+\frac{3}{2},\frac{e}{2},\frac{e}{2}+\frac{1}{2};\end{matrix}\quad -x^2\right]+\frac{x^2}{e(e+1)}\,{}_0F_1\left[\begin{matrix}-\;; \\ v+2;\end{matrix}\; -x^2\right]$$

$$+\frac{(e+2)x}{(v+2)e}\,{}_3F_4\left[\begin{matrix}\frac{v}{2}+1,\frac{e}{2}+2,\frac{e}{2}+\frac{3}{2}; \\ v+2,\frac{v}{2}+2,\frac{e}{2}+1,\frac{e}{2}+\frac{1}{2};\end{matrix}\quad -x^2\right]$$

$$-\frac{2x}{e}\,{}_1F_2\left[\begin{matrix}\frac{e}{2}+\frac{3}{2}; \\ v+2,\frac{e}{2}+\frac{1}{2};\end{matrix}\quad -x^2\right]-\frac{x^3}{(e+1)e(v+2)}\,{}_0F_1\left[\begin{matrix}-\;; \\ v+3;\end{matrix}\; -x^2\right]$$

Where,

g = $\frac{en}{v+1+n-e}$ and g' = $\frac{(e+1)n}{v+1+n-e}$

Which completes the proof of theorem 1

Special case:

when e = v+1, g = (v+1) and g' = (v+2)

$$e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{(1+v)_n}\,L_n^{(v+1)}(x)$$
$$=\Gamma(v+1)x^{-v-1}\{(x+v+1)J_{v+1}(2x)-\;xJ_{v+2}(2x)\} \qquad (3.5)$$

Replacing v+1 by v, result is

$$e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{(v)_n}\,L_n^{(v)}(x)=\Gamma(v)x^{1-v}\{J_{v-1}(2x)+\;xJ_v(2x)\} \qquad (3.6)$$

Which is a well-known result obtained by Kim et al. [Eq. 4.5, 9].

If v = ±½ in above result, special summation is obtained in terms of trigonometric functions. Kim et al. [9]

Thm 2

References:

1. Gurmeet Singh, Jyoti Garg, Coefficient inequality for A New Class of Analytic Functions; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 4 Issue 1 (2015), Pg 308-310
2. Bailey, WN: Generalized Hypergeometric Series, Cambride University Press, Cambridge (1935) Reprinted by Stechert-Hafner,New York (1964)
3. Eldelyi, A: Higher Transcendental Functions, McGraw-Hill, New York (1953)
4. Exton , H: Multiple Hypergeometric Function, Halsted, New York (1976)

5. Exton, H: Summation formulae involving the Laguerre polynomials. J. Comput. Appl. Math. . **100,** 225-227 (1998).
6. Kim, YS, Rakha, MA, Rathie, AK: Extension of certain classical summation theorems for the series $_2F_1$, $_3F_2$, and $_4F_3$ with applications in Ramanujan's summations. Int. J. Math. Sci. . **2010**, Article ID 309503 (2010)
7. Dr. K. Chithra, Edge - Odd Graceful Labeling of the Windmill Graphs; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 3 Issue 2 (2014), Pg 820-821
8. Kim, YS, Rakha, MA, Rathie, AK: Extension of certain classical summation theorems for the series $_2F_1$, $_3F_2$, and $_4F_3$ with applications in Ramanujan's summations. Int. J. Math. Sci. . **2010**, Article ID 309503 (2010)
9. Kim, YS, Rathie, AK, Paris, RB: On a new class of summation formulae involving the Laguerre polynomial. Integral Transforms Spec. Funct. . **23** (6), 435-444 (2012).
10. T. Jai Sankar, Vijayalakshmi.C ,Implementation of Stochastic Trend Analysis Model for Predicting the Fertilizers Consumption; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 4 Issue 1 (2015), Pg 301-307
11. Prudnikov, AP, Brychkov, YA, Marichev, OI: Integrals and series, Volume 3: More special functions, Nauka, Moscow (1986) (in Russian). Transleted from the Russian by GG Gould, Gordon & Breach, New York (1990)
12. Rainville, ED: Special Function, Macmillan Co., New York (1960) Reprinted by Chelsea, New York (1971)
13. Christopher Jefferson Y, M.Chandramouleeswaran, Fuzzy Algebraic Structure In Bp – Algebras; Mathematical Sciences International Research Journal : ISSN 2278-8697Volume 4 Issue 2 (2015), Pg 336-339
14. Rakha, MA, Rathie, AK: Generalization s of classical summation theorems for the series $_2F_1$ and $_3F_2$ with appilications. Integral Transform Spec. Funct. . **22** (11), 823-840 (2011).
15. Slater, LJ: : Generalized Hypergeometric Function, Cambridge University Press, Cambridge (1966)
16. Rakha, Medhat A., and Arjun K. Rathie. "Generalizations of classical summation theorems for the series 2 F 1 and 3 F 2 with applications." *Integral Transforms and Special Functions* 22.11 (2011): 823-840.
17. Rathie, Arjun K., and Adem Kılıçman. "On a new class of summation formulas involving the generalized hypergeometric F22 polynomial." *Advances in Difference Equations* 2014.1 (2014): 43.
18. Rama Krishna Majji, Satishkumar Ravupalli, the Standard Model for the Evolution of the Universal Frame; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 4 Issue 1 (2015), Pg 290-292
19. Kim, Y. S., A. K. Rathie, and Richard B. Paris. "On a new class of summation formulae involving the Laguerre polynomial." *Integral Transforms and Special Functions* 23.6 (2012): 435-444.
20. Paris, R. B. "A Kummer-type transformation for a 2F2 hypergeometric function." *Journal of computational and applied mathematics* 173.2 (2005): 379-382.
21. Rashi Bansal, Khushbu Rani, Coefficient inequality for A New Subclass of A Class of Starlike As Well As Convex Functions; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 4 Issue 1 (2015), Pg 298-300
22. Kim, Yong Sup, Medhat A. Rakha, and Arjun K. Rathie. "Extensions of Certain Classical Summation Theorems for the Series 2 *F* 1, 3 *F* 2, and 4 *F* 3 with Applications in Ramanujan's Summations." *International Journal of Mathematics and Mathematical Sciences* 2010 (2010).
23. Dr Anil Kumar, Aarti Ratra, Analysis of Pulsatile Flow of An Elastico-Viscous Fluid; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 3 Issue 1 (2014), Pg 312-319
24. Prudnikov, A. P., Yu A. Brychkov, and O. I. Marichev. "Integrals and series: special functions, Gordon and Breach Sci." *Publ., New York-London* (1986).
25. Rainville, Earl David. *Special functions*. New York, 1960.
26. Geethamalini.S, Balamuralitharan.S, Estimation of the Basic Reproductive Number; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 3 Issue 2 (2014), Pg 813-819

***

Yashoverdhan Vyas/ Department of Mathematics/ School of Engineering/
Kalpana Fatawat/ Sir Padampat Singhania University/ Udaipur/ Rajasthan/