A STUDY ON NET PROFIT ASSOCIATED WITH QUEUEING SYSTEM SUBJECT TO CATASTROPHICAL EVENTS

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Abstract: In this paper we study that the catastrophic events arrive independently at the service facility according to a Poisson process with rateγ. The nature of a catastrophic event is that upon its arrival at a service station, it destroys all the customers there waiting and in the service. We will derive the net profit associated with queueing system and obtain its probability of the busy period.

Keywords: Queueing system, Net-profit, busy period, Catastrophical events

Introduction: In the analysis of some queueing system, we come across situations where the annihilation of all the system and paralysation of the service facility may take place upon the arrival of some kind of special events. These special events are called catastrophical events and they themselves form a point process which may be independent of the arrival and service pattern of the queueing system. Such events occur quite commonly in computer networks. for example, When an infected job or file arrive at a service station, the job or the file acts as a catastrophic event destroying all the files in the processor and paralysing momentarily the processor. Models with customers impatience in queues have been studied by varies researchers. Krishna Kumar and Arivudainambi (2000) have analysed transient solution M /M/ 1 queue with catastrophes. The expression for the probability of the server being ideal, mean queue size and steady state probabilities are derived. Krishna Kumar and Pavaimadheswari (2002) have analysed two server queueing model with transient behaviour. The presence of catastrophes in the ongoing service and the feature effects are discussed. The source of impatience either a long Waite already experience in the queue or a long Waite anticipated by a customer on arrival. Altman and Yechiali (2005) and (2006) have analysed with customers impatiences when the server on vacation and unavailable for service. Sudhesh (2010) have argued that the queueing models with disasters team to be appropriated in some computer network or telecommunications applications and derived transient solution of a single server queue with system disaster and customer impatience. Chandrasekaran and Saravanarajan (2012) have used the continued fraction technique to obtain the transient solution of the M/ M/ 1 queue with feedback subject to catastrophe, server failures and repairs. Nompazy and Yechiali (2014) have studied an M/M/1 queue in a multi-phase random environment, where the system suffers a disastrous failure, causing on present job to be lost. The system at once moves to repair phase. After the repair is over it moves to ith phase. The probability generating methods have been applied to study the behaviour of the system. The transient analysis of queuing system demands methodologically simple and easily numerically implementable approach. In this content, we propose simple and mathematically elegant method of obtaining net profit associated with catastrophes events.

The rest of the paper is organized as follows: In the next section, we describe the mathematical model. Section 3 provides Analytic methods of the busy period analysis are given in Section 4. Section 5 investigates the busy period of our model. Finally, we will derive the net profit associated with queueing system and obtain its probability of the busy period.

Model Description and Analysis: We consider the M/M/1 queue with the possibility of catastrophes. Customers arrive at a single server facility. Following an orderly stationary Poisson stream with rate λ. The service time distribution is exponential with parameter μ and service discipline is first come first serve. Let $\{X(t); t\,\varepsilon R^{+}\}$ be the number of customers in the system at time t.

Let $P_n(t) = P(X(t) = n), n = 0,1,2...$ denote the transient state probability that there are n customers in the system, $P(s,t) = \sum_{n=0}^{\infty} P_n(t) s^n$ its probability generating function and m(t) its mean. From the above assumptions.

Let $P_n(t)$ be the probability that there are n customers in the system at time t. Then by the routine procedure, we have

$$P_n'(t) = \mu p_{n+1}(t) - (\lambda + \mu + \gamma) p_n(t) + \lambda p_{n-1}(t), n = 1,2 \ldots .... \quad (1.1)$$

$$P_0'(t) = \mu p_1(t) - \lambda p_0(t) + \gamma[1 - P_0(t)]$$

Where λ and μ have the ususal meanings. We assume that a customer arrives to an empty system at the service facitity at time t=0 so that busy period starts at time t= 0.

Then $_n(t) = \delta_{ij}$, $n = 0,1,2...$ where

$$\delta_{ij} = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases}$$

There are different methods to approximate the solution of the above system. We use the generating function technique to derive the solution of the above system in the following section. To solve (1.1) We proceed as follows, Defining

$$P(s,t) = \sum_{n=0}^{\infty} p_n(t)\, s^n$$

We obtain from (1.2)

$$\frac{\partial P(s,t)}{\partial t} = \left[\lambda s + \frac{\mu}{s} - (\lambda + \mu + \gamma)\right] P(s,t) + \mu\left(1 - \frac{1}{s}\right) p_0(t) + \gamma \qquad (1.2)$$

Subject to the condition (s,0)=s . The equation (1.2) can be solved and we obtain

$$P(s,t) = s\, e^{-(\lambda+\mu+\gamma)t} e^{\left(\lambda s + \frac{\mu}{s}\right)t} + \mu\left(1 - \frac{1}{s}\right)\int_0^t p_0(u)\, e^{-(\lambda+\mu+\gamma)(t-u)} e^{\left(\lambda s+\frac{\mu}{s}\right)(t-u)} du + \gamma \int_0^t e^{\left(\lambda s+\frac{\mu}{s}\right)(t-u)} du \qquad (1.3)$$

In (1.3) we use the integrating function

$$exp\left\{\left(\frac{\mu}{s} + \lambda s\right) t\right\} = exp\left\{\frac{1}{2}\left((\beta s) + \frac{1}{(\beta s)}\right)(\alpha t)\right\} = \sum_{-\infty}^{\infty} (\beta s)^n I_n(\alpha t) \qquad (1.4)$$

Where we have set λ=αβ/2 and μ=α/2β we have α=√(2&λμ) and =√(λ/μ) . In the above are modified Bessel functions of the first kind given by

$$I_n(u) = \sum_{k=0}^{\infty} \frac{u^{n+2k}}{2^{n+2k} k!\,(n+k)!}, n > -1, I_{-n}(u) = I_n(u)$$

$$= I_n(u)$$

The equating the powers of s^n on bothsides,

$$P_0(t) = \frac{1}{\mu}\sum_{k=1}^{\infty} \frac{(n+1)}{\beta^{n+1}} \frac{I_n(\alpha t)}{t}\, e^{-(\lambda+\mu+\gamma)t} + \frac{\gamma}{\mu}\int_0^t \frac{n}{\beta^n}\frac{I_n(\alpha u)}{u} e^{-(\lambda+\mu+\gamma)u} du$$

$$P_n(t) = \frac{2\gamma\beta^{n+1}}{\alpha}$$

$$X \int_0^t \sum_{k=0}^{\infty} \frac{(n+k+1) I_{n+k+1}(\alpha u)}{\beta^{k+1}\mu}\; e^{-(\lambda+\mu+\gamma)u} du$$

$$P_n(t) = \frac{2\gamma\beta^{n+1}}{\alpha}$$

$$X \int_0^t \sum_{k=0}^{\infty} \frac{(n+k+1) I_{n+k+1}(\alpha u)}{\beta^{k+1}\mu}\; e^{-(\lambda+\mu+\gamma)u} du + \sum_{m=0}^{\infty} e^{-(\lambda+\mu+\gamma)t}\left[\frac{I_{m+n+2}(\alpha t)}{\beta^{m-n+2}} - \frac{I_{m+n+3}(\alpha t)}{\beta^{m-n+1}}\right] + \beta^{n-1} I_{n-1}(\alpha t) e^{-(\lambda+\mu+\gamma)t}, \quad n = 1,2,.$$

The above probabilities completely describe the queueing process.

Analytic Method of The Busy Period Analysis: We have already mentioned that the busy and the idle periods develop a random evolution in problems related to the queueing systems. We proceed to obtain the probability law of the busy period [7]. To do this, we impose further that there is an absorbing barrier at zero system size so that $P_0'(t)$ gives the probability density function of the busy period [8], where $P_n(t)$ represents the probability that the system size at time **t** is **n .** We assume that the server enters into the busy period at time t = 0. Then $P_1(t) = 1, P_n(t) = 0$ for n≠1 with absorption at the state 0 are

$$P_0'(t) = \mu P_1(t) + \gamma[1 - P_0(t)] \qquad (3.1)$$

$$P_n(t) = -(\lambda + \mu) P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t), n \geq 1 \qquad (3.2)$$

Equations (3.1) and (3.2) are subject to the condition $P_n(0) = \delta_{1,n}\ n = 0,1,2..$ it is clear that $P_0'(t)$ is the probability density function of the busy period. To find it we proceed as follows: Define

$$K(s,t) = \sum_{n=0}^{\infty} P_n(t) s^n$$

Then K(S,0) = S and

$$\frac{\partial K(s,t)}{\partial t} = -\left(\lambda + \mu + \gamma - \lambda s - \frac{\mu}{s}\right) K(s,t) + \left(\lambda + \mu - \frac{\mu}{s}\right) p_0(t) + \gamma. \qquad (3.3)$$

Integrating (3.3) we get

$$K(s,t) = s e^{-(\lambda+\mu+\gamma)t} e^{\left(\lambda s + \frac{\mu}{s}\right)t}$$

$$+\left(\lambda+\mu-\frac{\mu}{s}\right)\int_0^t e^{-(\lambda+\mu+\gamma)(t-u)}e^{\left(\lambda s+\frac{\mu}{s}\right)(t-u)}p_0(u)du$$

$$+\lambda\int_0^t e^{-(\lambda+\mu+\gamma)(t-u)}e^{\left(\lambda s+\frac{\mu}{s}\right)(t-u)}du. \quad (3.4)$$

Substituting the series expressions for K(s,t) and $e^{\left(\lambda s+\frac{\mu}{s}\right)t}$ into (3.4) and equating the coefficients of s^o on both sides, we get

$$p_0(t)=\frac{I_1(\alpha t)}{\beta}e^{-(\lambda+\mu+\gamma)t}$$

$$+(\lambda+\mu)\int_0^t e^{-(\lambda+\mu+\gamma)(t-u)}I_0(\alpha(t-u))p_0(u)du$$

$$-\mu\beta\int_0^t e^{-(\lambda+\mu+\gamma)(t-u)}I_1(\alpha(t-u))p_0(u)du$$

$$+\gamma\int_0^t e^{-(\lambda+\mu+\gamma)(t-u)}I_0(\alpha(t-u))du. \quad (3.5)$$

The integral (3.5) admits a Laplace transform solution for $P_0(t)$. If $P_0'(t)$ is the Laplace transform of $P_0(t)$ and $I_0^*(t)$ is the Laplace transform of hen we obtain

$$p_0^*(\theta)=\frac{1}{\beta}\left[\frac{\beta\gamma I_0^*(\frac{\theta+\lambda+\mu+\gamma}{\alpha})+I_1^*((\frac{\theta+\lambda+\mu+\gamma}{\alpha})}{\alpha-(\lambda+\mu)I_0^*(\frac{\theta+\lambda+\mu+\gamma}{\alpha})+\beta\mu I_1^*((\frac{\theta+\lambda+\mu+\gamma}{\alpha})}\right] \quad (3.6)$$

Inverting (3.6), we obtain the derivative $P_0'(t)$ and this gives the probability density function of the busy period.

Busy Period Analysis: A busy period is defined as the interval of time commencing at the instant o when a customer arrives at an empty counter and terminating at the instant when the server becomes free for the first time. Let the length of the interval which is a random variable be T and $\{N^*(t)\}$ be the stochastic process denoting the number of customers present at the instant t during the busy period.

We have $\{N^*(t)=1\}$ and the duration of the busy period is the first t for which

Now $q_n(t)$ will satisfy the following difference differential equations

$$q_1'(t) = -(\lambda+\mu q+\gamma)q_1(t) + \mu q q_2(t) \quad (4.1)$$

$$q_n'(t) = -\lambda q_{n-1}(t)-(\lambda+\mu q+\gamma)q_n(t) + \mu q q_{n+1}(t), n = 2,3 \ldots \quad (4.2)$$

We solve the above equations using generating function method. By similar arguments as in section 2, the solution for the above equations is given by

$$q_n(t) = \left(\frac{\lambda}{\mu q}\right)^{\frac{n}{2}}\frac{n}{\lambda t}e^{-(\lambda+\mu q+\gamma)t}I_n\left(2t\sqrt{\lambda\mu q}\right)n = 1,2,\ldots$$

Conditioning on the number of customers present at instant t, all of their services in (t, t+dt) we have

$$b(t)dt \cong P\{t\le T<t+dt\}$$

$$=\sum_{j=1}^{\infty}P\{t\le T<t+dt \mid N^*(t)=j\}\,P\{N^*(t)=1\}$$

$$=P\{t\le T<t+dt\mid N^*(t)=1\}\,P\{N^*(t)=1\}$$

$$+\sum_{j=2}^{\infty}P\{t\le T<t+dt\mid N^*(t)=j\}\,P\{N^*(t)=j\}$$

The first term implies that there is only one customer T the instant t whose service is completed between (t, t+dt), the probabilities of this event being $\mu q dt+0(dt)$. The second term implies service completion of two or more customers (t, t+dt) in and the probability of this event is $0(dt)$. Thus taking the limit as $dt\to 0$.

$$b(t) = [\mu q q_1(t)]$$

$$=\left(\frac{\lambda}{\mu q}\right)^{\frac{1}{2}}\frac{1}{\lambda t}e^{-(\lambda+\mu q+\gamma)t}I_1\left(2t\sqrt{\lambda\mu q}\right) \quad (4.3)$$

The Laplace transform of T is given by

$$b^*(t)=L\{b(t)\}=\mu q L\{q_1(t)\}= \mu q q_1^*(z)$$

$$=\left[\frac{(z+\lambda+\mu q+\gamma)}{2\lambda}\right]+\left[\frac{-\sqrt{(z+\lambda+\mu q+\gamma)^2-4\lambda\mu q}}{2\lambda}\right]$$

Net-Profit of Random Evolution Of Stochastic Integral: In this proposed project the idle period occurs whenever a catastrophic event occurs when server is busy [2]. Let there be a positive income when the server is busy, and a cost to pay when it is idle[6]. To study the net gain, we define the following costs. Let C_1 be the profit per unit time of the busy period, C_2 be the cost per unit time of the idle period not initiated by the departure of a catastrophic event and C_3 be the cost per unit time of the idle period initiated by the departure of a catastrophic event[9]. Then the time – course of the net profit can be described by the random motion of a stochastic integral. To achieve this, we define stochastic process Z(t) as:

$$Z(t)=\begin{cases}1 \text{ if the server is busy}\\ 2 \text{ If the server is idle initated by Catastrophe}\\ 3 \text{ If the server is not idle initated by Catastrophe}\end{cases}$$

The stochastic process Z(t) is a market- point process and its probability law can be obtained in terms of the distributions of the busy and idle periods[4]. The idle periods are of two types and are characterized by the point process of catastrophic events[5], [10]. We not that

$$P_r\{Z(t)=1\}=\sum_{n=1}^{\infty}\int_0^t P_n(u)\,e^{-(\alpha+\lambda+\mu)(t-u)}du$$

$$P_r\{Z(t)=2\}=\int_0^t P_1(u)\,e^{-\lambda(t-u)}du$$

$$P_r\{Z(t)=3\}=\sum_{n=1}^{\infty}\int_0^t P_1(u)\,\gamma\,e^{-\lambda(t-u)}du$$

If C(t) is the instantaneous cost at time t then

$$C(t)=\begin{cases}C_1, if\ z(t)=1\\ C_2, if\ z(t)=2\\ C_3, if\ z(t)=3\end{cases}$$

Then the net gain X(t) is given by $X(t)=\int_0^t C(u)du$ is identified as the position of the particle and the probability law of X(t) can be obtained by considering the random motion of the particle.

Conclusion: In this paper, a single server queue in which the inter arrival times follow exponential distribution and service times follow Poisson distribution has been considered. If the catastrophe occurs according to Poisson processes then we study the net profit. We obtained the busy period analysis of the random evolution of a stochastic integral with net-profit C(t) as instantaneous cost at time t. For this model, through probability generating functions, the expected queue size and variance under the system is down as well as that is functioning have been derived.

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