A DIRECT COLE-HOPF TRANSFORMATION OF A GENERALIZED BURGERS EQUATION WITH VARIABLE VISCOSITY TO SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATION

B. MAYILVAGANAN, S.V. GOMATHY

Abstract: Invariant solutions of a generalized Burgers equation with variable viscosity are obtained by non-classical Lie group method. To the best of our knowledge we are first to report the linearization of the generalized Burgers equation with variable viscosity.

Keywords: Cole-Hopf Transformation, Generalized Burgers equation, Linear Equation.

Introduction: The Burgers equation for u(x, t) is [1]

$$u_t + u\,u_x = \frac{1}{2}\,u_{xx} \quad (1)$$

Hopf [2] and Cole [3] have shown that the second order nonlinear "partial differential equation (PDE)" (1) may be directly transformed to the second order linear PDE

$$\phi_t = \frac{\delta}{2}\,\phi_{xx} \quad (2)$$

through the Cole-Hopf transformation

$$u(\mathrm{x},\mathrm{t}) = \frac{1}{\phi(\mathrm{x},\mathrm{t})}\,\frac{\partial\phi(\mathrm{x},\mathrm{t})}{\partial \mathrm{x}} \quad (3)$$

New nonlinear parabolic and hyperbolic equations for u(x, t) were generated from their respective counterparts, viz.,

$$\phi_t + b\,\phi_x + \mathrm{c}\,\phi + \mathrm{f} = \epsilon\,\mathrm{a}\,\phi_{xx} \quad (4)$$

$$\phi_t + b\,\phi_x + \mathrm{c}\,\phi + \mathrm{f} = \epsilon\,\mathrm{a}\,\phi_{xt} \quad (5)$$

where a, b, c, f are functions of x, t, and ϵ is a parameter, by Sachdev [4] from the more general Cole-Hopf transformations in the form

$$F(u) = k(x,t)(\log\phi)_x \quad (6)$$

Nimmo and Crighton [5] have derived B$\ddot{a}$cklund transformations between two nonlinear parabolic equations of the form

$$u_t - u_{xx} + H(t,x,u,u_x) = 0 \quad (7)$$

And concluded that besides the Burgers equation [1] its inhomogeneous version, viz.,

$$u_t + u\,u_x + \mathrm{a}(\mathrm{x},\mathrm{t}) = \frac{\delta}{2}\,u_{xx} \quad (8)$$

is also linearisable.

Kumei and Bluman [6] have devised a method to determine necessary and sufficient conditions under which a system of nonlinear equations with n ≥ 2 independent variables and m ≥ 1 dependent variables may be linearized. Their work contained an algorithm to explicitly construct an invertible mapping μ, if exists, between the systems of nonlinear and linear PDEs.

Bluman [7] presented an algorithm which maps linear partial differential equations (PDEs) with variable coefficients to linear PDEs with constant coefficients. Later, Sachdev and Mayil Vaganan [8], using Clairin's method (Clairin [9,10] as detailed in Lamb [11,12]), derived generalised B□cklund transformations involving derivatives upto any finite order for linear parabolic and hyperbolic PDEs with variable coefficients.

Kawamoto [13] derived certain nonlinear evolution equations reducible to the Painlev$\acute{e}$ equations by applying Lie's Classical Method (Bluman and Anco [14] and Olver [15]). Later, Sachdev and B. Mayil Vaganan [16] derived several mappings between the solutions of nonlinear evolution equations like the modified Burgers equation, the cylindrical K-dV equation etc.

In this paper, we extend the work of Mayil Vaganan and Senthilkumaran [17]; Indeed, we shall carry out the following:

- Self-similar solution form is identified for the GBE $u_t + u^{n-1}\mathrm{u}_\mathrm{x} = \frac{1}{n}\, t^{\frac{2}{n}-1} u_{xx}$, $\mathrm{n} \in \mathrm{Z}^+$ (9) in order to reduce it to an ODE (Note that the PDEs (9) and (1) are identical when n = 2)
- Derive the following Cole-Hopf transformation to (9):

$$\mathrm{u}(\mathrm{x},\mathrm{t}) = \mathrm{R}\ \mathrm{t}^{-\frac{1}{\mathrm{n}}}\left[\frac{1}{\phi(\mathrm{z})}\,\frac{\mathrm{d}\phi(\mathrm{z})}{\mathrm{dz}}\right]^{1/(n-1)},$$

$$\mathrm{z} = \mathrm{xt}^{-\frac{1}{\mathrm{n}}},\ \mathrm{R} = \left[\frac{1}{1-\mathrm{n}}\right]^{1/(\mathrm{n}-1)} \quad (10)$$

Using (10), we transform (9) directly (unlike in the work of Mayil Vaganan and Senthilkumaran [17]) to the following first order linear "ordinary differential equation (ODE)" for φ (z):

$$\varphi'(\mathrm{z}) + (\mathrm{n}-1)\mathrm{z}\varphi(\mathrm{z}) = 0,$$

$$\varphi(z) = \frac{\mathrm{d}\phi(\mathrm{z})}{\mathrm{dz}} \quad (11)$$

Three Steps Transformation of (9) to Linear Equation:

We shall now transform the GBE (9) directly to a second order linear ODE (11) through Cole-Hopf transformation. For, we seek solutions of (9) in the form

$$u(x,t) = t^A f(z),\ z(x,t) = x\,t^B \quad (12)$$

where A and B are to be determined. Substitution of (12) into the GBE (9) results in the nonlinear second order ODE

$$f'' - nf^{n-1}f' + zf' + f = 0 \quad (13)$$

provided that A = B = - 1/n. Equations in (12) are rewritten as

$$\mathrm{u}(\mathrm{x},\mathrm{t}) = t^{-1/n}\ \mathrm{f}(\mathrm{z}) \quad (14)$$

$$\mathrm{z}(\mathrm{x},\mathrm{t}) = \mathrm{x}\ t^{-1/n} \quad (15)$$

It is easily verified that the equation (13) may be put as the derivative of the following Bernoulli's equation, viz., as a "first integral"

$$\frac{d}{dz}[f' - f^n + zf] = 0 \qquad (16)$$

Equation (16) translates into the fact that the Bernoulli's equation

$$f' - f^n + z\,f = 0 \qquad (17)$$

is, indeed, a "first integral" of the f-equation (13). In doing so, the constant of integration is set equal to zero.

As expected, the Bernoulli's equation (17) may be linearized to the Lagrange's equation

$$H' - (n-1)zH + n - 1 = 0 \qquad (18)$$

Via the transformation

$$H(z) = [f(z)]^{1-n} \qquad (19)$$

Surprizingly the transformation

$$H(z) = [f(z)]^{-(1-n)} \qquad (20)$$

which, in a sense, is an "inverse" to (19) replaces the n-Bernoulli's equation (17) by a 2-Bernoulli's equation

$$H' - (n-1)H^2 + (n-1)z\,H = 0 \qquad (21)$$

The transformation(20) changes the f-equation (13) to an Euler- Painlevé transcendent

$$H\,H'' - \frac{(n-2)}{(n-1)}H'^2 - n\,H^2\,H' + (n-1)H^2 + z\,HH' = 0 \qquad (22)$$

Interestingly, there is both the "linear" and "nonlinear" first integrals for the Euler- Painlevé Transcendent (22), namely, the Lagrange's equation (18) and the Riccati equation (21).

It is remarkable that both the two different transformations (19) and (20) transform the f-equation (13) to the "same" Euler- Painlevé equation (22).

It is a well known fact that a Riccati equation (2-Bernoulli's equation) can be linearized to a second order ODE. Indeed, the transformation

$$H(z) = \frac{1}{1-n}\frac{1}{\phi(z)}\frac{d\phi(z)}{dz} \qquad (23)$$

linearize the Riccati equation (21) to the first order linear ODE for $\varphi(z)$:

$$\varphi'(z) + (n-1)z\,\varphi(z) = 0, \varphi(z) = \frac{d\phi(z)}{dz} \qquad (24)$$

We now summarize:

- Transformed the GBE with power law viscosity (9), using the similarity transformation (14) - (15) to the second order nonlinear ODE (13) .
- Equation (13) is directly integrated to yield a first integral (17) which is a Bernoulli's equation
- The intermediate integral (17) is transformed to the Riccati equation (21) via the nonlinear transformation (20)
- Riccati equation (21) is transformed via the Cole-Hopf transformation (23) to the second order linear ODE (24) for $\phi(z)$.

Direct Transformation of (9) to the Same Linear Equation (24): Inserting for H(z) from (23) into the nonlinear transformation (20) yields

$$[f(z)]^{n-1} = \frac{1}{n-1}\frac{1}{\phi(z)}\frac{d\phi(z)}{dz} \qquad (25)$$

Or

$$f(z = \left(\frac{1}{n-1}\right)^{1/(n-1)}\left[\frac{1}{\phi(z)}\frac{d\phi(z)}{dz}\right]^{1/(n-1)} \qquad (26)$$

If we replace for f(z) from (26) in (14)-(15), the

$$u(x,t) = R\ t^{\frac{-1}{n}}\left[\frac{1}{\phi(z)}\frac{d\phi(z)}{dz}\right]^{\frac{1}{n-1}} \qquad (27)$$

$$z(x,t) = x\ t^{\frac{-1}{n}} \qquad (28)$$

where we used

$$R = \left(\frac{1}{n-1}\right)^{1/(n-1)} \qquad (29)$$

Now inserting from (27) and (28), the GBE (9) becomes

$$\frac{-R}{n}t^{-1}\,z\,\phi^{\frac{-1}{n-1}}(\phi')^{\frac{1}{n-1}} + \frac{R^n}{n}t^{-1}\phi^{-n/(n-1)}(\phi')^{n/(n-1)}$$

$$= \frac{R}{n}t^{-1}\left(\frac{-1}{n-1}\phi^{-n/(n-1)}(\phi')^{n/(n-1)} + \frac{1}{n-1}\phi^{(2-n)/(n-1)}\phi''\right) \qquad (30)$$

It is interesting to note from (29) that

$$R^{n-1} = \left[\left(\frac{1}{n-1}\right)^{1/(n-1)}\right]^{n-1} = \frac{1}{1-n} \qquad (31)$$

In view of (31), the second term on the LHS coincides with the first term on the RHS (30), and we thus have

$$\frac{-R}{n}t^{-1}\,z\,\phi^{-1/(n-1)}(\phi')^{1/(n-1)}$$

$$= \frac{R}{n}t^{-1}\frac{1}{n-1}\phi^{\frac{-1}{n-1}}(\phi')^{\frac{2-n}{n-1}}\phi'' \qquad (32)$$

Which simplifies to the second order linear equation (24).

References:

1. Dr.K.Chithra, S.Vanitha, Properties of Null-Additive Fuzzy Measure on Locally; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 3 Issue 2 (2014), Pg 876-877
2. J. M. Burgers, in: R. von Mises, T. von Karman, (Eds), A Mathematical Model Illustrating the Theory of Turbulence, Adv. Appl. Mach. 1: 171-199 (1948).
3. E. Hopf, The partial differential equation $u_t + u\,u_x = u_{xx}$Commun. Pure Appl. Math. 3 201-230 (1950).
4. J. D. Cole, On a quasi-linear parabolic equation occurring in aerodynamics, Quart. Appl. Math. 9 225{236 (1951).
5. P. L. Sachdev, A generalised Cole-Hopf transformation for nonlinear parabolic and hyperbolic equations, ZAMP 29 963-970 (1978).
6. J. J. C. Nimmo and D. G. Crighton, Bäcklund transformations for nonlinear parabolic equations: The general results, Proc. R. Soc. Lond. A **384** 381-

401 (1982).
7. S. Kumei and G. W. Bluman, When nonlinear differential equations are equivalent to linear differential equations, SIAM J. Appl. Math. 42 1157-1173 (1982).
8. G. Mahadevan, V. G. Bhagavathi Ammal, C.Sivagnanam, Restrained Triple Connected Domination Number And Connectivity Of Graphs; Mathematical Sciences International Research Journal : ISSN 2278-8697Volume 4 Issue 2 (2015), Pg 281-283
9. G. W. Bluman, On mapping partial differential equations to constant coeffcient equations, SIAM J. Appl. Math. 43 1259-1273 (1983).
10. P. L. Sachdev and B. Mayil Vaganan, Exact Solutions of Linear Partial Differential Equations with Variable Coeffcients, Stud. Appl. Math. 87:213-237 (1992).
11. J. Clairin, Sur les transformations de Bäcklund, Ann. Sci. Ecole Norm. Sup. 3e Ser. Suppl. 19 s1-s63 (1902).
12. K.Padmapriya, Dr.S.Sridhar, Group Nearest Neighbor Queries Against Data Piracy; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 3 Issue 2 (2014), Pg 878-882
13. J. Clairin, Sur quelques equations aux derivees partielles du second ordre, Ann. Fac. Sci. Univ. Toulouse 2e Ser. 5 437-458 (1903).
14. G. L. Lamb, Jr, Bäcklund transformations for certain nonlinear evolution equations, J. Math. Phys. 15 2157-2165 (1974).
15. G. L. Lamb, Jr, Bäcklund transformations at the turn of the century, "Bäcklund transformations, the inverse scattering method, solitons, and their applications," Lecture Notes in Mathematics, 515, (Ed. R. M. Miura), pp. 69{79, Springer-Verlag, Berlin, 1976.
16. S. Kawamoto, Derivation of nonlinear partial differential equations reducible to the Painlevé equations, J. Phys. Sco. Jpn. 52 (1983) 4059-4065.
17. P. L. Sachdev and B. Mayil Vaganan, On the mapping of solutions of nonlinear partial differential equations, Nonlin. World 2 (1995) 171-189.
18. M.P.Singh, on Frechet Spaces of Distributions; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 3 Issue 2 (2014), Pg 893-898
19. M. J. Lighthill, Viscosity effects in sound waves of finite amplitude. In Surveys in Mechanics (Eds: G. K. Batchelor and R.M. Davies), 250-351 (1956).
20. J. Doyle and J. Englefield, Similarity solutions of a generalized Burgers equation, IMA J. Appl. Math. **44** 145-153 (1990).
21. B. Mayil Vaganan and M. Senthil Kumaran, Exact Linearization and Invariant solutions of a Generalized Burgers equation with Linear Damping and Variable Viscosity, Stud. Appl. Math. 117: 95 - 108 (2006).
22. Fahad Al Basir , Sumit Nandi , Priti Kumar Roy, Optimal Control on Saponification to Maximize; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 3 Issue 1 (2014), Pg 441-446
23. B. Mayil Vaganana and M. Senthil Kumaran, Kummer Function Solutions of Linearly Damped Burgers Equations with Time-dependent Viscosity by Exact Linearization, Nonlinear Analysis: Real World Applications 9: 2222 - 2233 (2008). doi:10.1016/j.nonrwa.2007.08.001
24. B. Mayil Vaganan and T. Jeyalakshmi, Generalized Burgers Equations Transformable to the Burgers Equation, Stud. Appl. Maths 127: 211 - 220 (2011). DOI: 10.1111/j.1467-9590.2010.000515.x
25. S. Lie, Klassiffkation und Integration von gewohnlichen Differentialgleichen zwischen x, y die eine Gruppe von Transformationen gestatten, Math. Ann. 32 213-281 (1988).
26. A. V. Bäcklund, Ueber Flächentransformationen, Math. Ann. 9 197-320 (1876).
27. Dr. Sanjeev Kumar, Bindu Batish, Optimal Fourth Order Variants of Ellipse Methods; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 4 Issue 1 (2015), Pg 220-223
28. B. Abraham-Shrauner and K. S. Govinder, Provenance of Type II hidden symmetries from nonlinear partial differential equations, J. Nonlinear Math. Phys. **13** 612 - 622 (2008).
29. M. L. Gandarias, Type-II hidden symmetries through weak symmetries for nonlinear partial differential equations, J. Math. Anal. Appl., 348 752{759 (2008).
30. M. P. Edwards, P. Broadbridge, Exceptional symmetry reductions of Burgers equation in two and three spatial dimensions, Z. Angew. Math. Phys. 46: 595-622 (1995).
31. M. A. Christou, N. M. Ivanova and C. Sophocleous, Similarity reductions of the (1+3)-dimensional Burgers equation, Appl. Math. Comp. 210: 87{99 (2009).
32. L. A. Richards, Capillary conduction of liquids through porous mediums, 1: 318 - 333 (1931)

***

B. Mayil Vaganan/ Professor & Head/ Department of Applied Mathematics & Statistics/
School of Mathematics/ Madurai Kamaraj University/ Madurai – 625021/ Tamil Nadu/ India/
S. V. Gomathy/ School of Mathematics/, Madurai Kamaraj University/Madurai – 625021/Tamil Nadu/ India/