MULTI OBJECTIVE FRACTIONAL PROGRAMMING UNDER (ρ , B) – CONVEXITY

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Abstract: Various generalizations of convex functions have appeared in the literature. In 1983 vial [48] developed the concept of convex function by introducing ρ-convex functions. The generalization of ρ-convex function ρ– psedudoconvex and ρ-quasiconvex functions was given by Jeyakumar [48]. A class of functions called b-vex functions were introduced by Bector and Singh [3]. In recent times some authors have defined more general class of functions labeled as (ρ , b) convex functions. This class is defined by relaxing the definitions of ρ-convex and b-vex functions. Namely certain examples for a wide variety of functions satisfying b-vexity functions can be seen from Bector et al [76] and Suneja and Lalitha [79]. For ρ-convex functions many references are available in the current literature. But they not consider in the recent developed concept like multi objective fractional programming under (ρ , b) – convexity. In this chapter we consider (ρ , b) – convexity and its generalizations and present some optimality conditions for a multiobjective fractional programming problem.

Keywords: ρ-convex, b-convex

Introduction: Various generalizations of convex functions have appeared in the literature. In 1983 vial [48] developed the concept of convex function by introducing ρ-convex functions. The generalization of ρ-convex function ρ– psedudoconvex and ρ-quasiconvex functions was given by Jeyakumar [48]. A class of functions called b-vex functions were introduced by Bector and Singh [3]. In recent times some authors have defined more general class of functions labeled as (ρ , b) convex functions. This class is defined by relaxing the definitions of ρ-convex and b-vex functions. Namely certain examples for a wide variety of functions satisfying b-vexity functions can be seen from Bector et al [76] and Suneja and Lalitha [79]. For ρ-convex functions many references are available in the current literature. But they not consider in the recent developed concept like multi objective fractional programming under (ρ , b) – convexity. In this chapter we consider (ρ , b) – convexity and its generalizations and present some optimality conditions for a multiobjective fractional programming problem.

Definitions Properties of (ρ , b) – convex function: For $x, y \in R^n$ by $x \le y$ we mean $x_i \le y_i$ for i=1,2,.... n by $x \le y$ we mean $x_i \le y_i$ for all i=1,2.... n and $x_i < y_i$ for at least on j, $i \le j \le n$ and for $x, y \in R$, x<y is used one j, $i \le j \le n$ and for $x, y \in R$, $x < y$ is used in usual sense. We shall now define that notion of (ρ , b) – convexity. Let $K = \{1, 2, ...k\}$ and $M = \{1, 2, ...m\}$

Definition: Let f be a numerical function defined on open set $S \subseteq R^n$ and let f be differentiable at $x \in s$. Let ρ be a real number and $\theta : SxS \to R^n$ and $b : x\,x\,s \to R_+$ be functions. Then the function f is said to be

(i) (ρ , b) – convex at $\bar{x}$ with respect of S, θ and b or briefly. (ρ , b) – convex at $\bar{x}$ is for each $x \in s$.

$$b\left(x,\bar{x}\right)\left[\frac{f_i\left(x\right)}{g_i\left(x\right)} - \frac{f_i\left(\bar{x}\right)}{g_i\left(\bar{x}\right)}\right] \ge \left(x-\bar{x}\right)^t \nabla \frac{f_i\left(\bar{x}\right)}{g_i\left(x\right)} + \rho \,\|\, \theta\left(x-\bar{x}\right) \|^2$$

(ii) (ρ , b) – pseudo convex at x if for each $x \in s$

$$b\left(x,\bar{x}\right)\left(x-\bar{x}\right)^t \nabla \frac{f_i\left(\bar{x}\right)}{g_i\left(\bar{x}\right)} + \rho \left\|\theta\left(x,\bar{x}\right)\right\|^2 \ge 0$$

Implies

$$\frac{f_i(x)}{g_i(x)} \geq \frac{f_i(\bar{x})}{g_i(\bar{x})}$$

(iii) (ρ , b) – quasi convex at $\bar{x}$ if for each $x \in S$

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(\bar{x})}{g_i(\bar{x})}$$

Implies $b(x,\bar{x})(x-\bar{x})^t \nabla \frac{f_i(\bar{x})}{g_i(\bar{x})} + p \| \theta(x,\bar{x}) \|^2 \leq 0$

Definition : A point $\bar{x} \in X$ is said to be efficient solution for (FP) if there does not exist any $x \in X$ such that $Q(x) \leq Q(\bar{x})$.

Definition : A point $\bar{x} \in x$ is said to be properly efficient for (FP) if it is efficient for (FP) and if there exists a scalar N>o such that for each i.

$$Q_i(\bar{x}) - Q_i(x) \leq N\left[Q_j(x) - Q_j(\bar{x})\right]$$

for some j such that $Q_j(x) > Q_j(\bar{x})$ when even x is feasible for (FP) and $Q_i(x) < Q_i(x) < Q_i(\bar{x})$

Formulation: The multiobjecitive fractional programming problem we consider in the following

(F.P) Minimize $\left\{ Q(x) = \frac{f_i(x)}{g_i(x)}; x \in X \right\}$

Subject to $h_j(x) \leq 0$; j=1,2,............m; $x \in S$

The feasible set $X = x \in S;\ h_j(x) \leq 0$; j=1,2.......m)

and $S \subseteq R^n$

Here $f_i : S \to R, g_i : S \to R;\ i = 1, 2, ...K$ and $h_j : S \to R,\ j = 1, 2, ...m$ are all assumed to be differentiable. It is further assumed that

(i) $g_i(x) > 0$ for all $x \in S$ and each i=1,2,..k;

(ii) $X \neq \phi$

Necessary ad Sufficient Condition: We consider the following parametric multiobjective optimization problem $(FP)_v$ for each $V \in R_+^K$ where R_+^K denotes the positive orthant of R^K $(FP)_v$ minimize $\left[f_i(x) - V_1 g_i(x), ...F_k(x) - V_k.g_k(x)\right]$

Subject to $h_j(x) \leq 0. j \in M : x \in S$

Lemma: We now have the following lemma connecting the properly efficient solutions of (FP) and $(FP)_v$.

Lemma: Let $\bar{x}$ be properly efficient for (FP). Then there exists $\bar{V} \in R_+^k$ such that $\bar{x}$ is properly efficient for $(FP)_{\bar{v}}$ conversely if $\bar{x}$ is properly efficient for $(FP)_{\bar{v}}$ where $\bar{V}_i = Q_i(\bar{x}), i \in K$ then $\bar{x}$ is properly efficient for (FP) following Geoffrion we now consider the scalar programme corresponding to $(FP)_{\bar{v}}$ given by

$$(FP)\frac{\lambda}{v} \text{ minimize } \sum_{i=1}^{k}\lambda_i\left[f_i(x)-\overline{V}_i\, g_i(x)\right]$$

Subject to $h_j(x)\le 0, j\in M, x\in S$

We have the following result on the lines of Geoffrion

Lemma: If $\overline{x}$ is an optimal solution for $(FP)\frac{\lambda}{v}$ for some $\lambda\in R^k$ with strictly positive components then $\overline{x}$ is properly efficient for (FP).

Under the assumption of (ρ , b) – convex, (ρ , b) – quasiconvex and (ρ , b) – Pseudo convex objective and constraint functions we now give a number of sufficient optimality criteria for a feasible $\overline{x}$ for (FP) to be properly efficient for (FP). These generalize the sufficient optimality criteria. Given by Suneta at et al.

Sufficient Optimality Condition:

Theorem: Let x be a feasible solution for (FP). Assume that there exists $\overline{V}$ (as given by Lemma , $\overline{\lambda}\in R_+^k$ and $\overline{\mu}_i\ge 0$ for each $j\in I=\left\{j:h_j\left(\overline{x}\right)=0)\right\}$ the set of indices of active constraint functions at $\overline{x}$, such that

$$\sum_{i=1}^{k}\overline{\lambda}_i\left[\nabla f_i\left(\overline{x}\right)-\overline{V}_i\nabla g_i\left(\overline{x}\right)\right]+\sum_{j\in I}\overline{\mu}_j\nabla h_j\left(\overline{x}\right)=0$$

Then $\overline{x}$ is properly efficient for (FP) if any of the following holds at $\overline{x}$ with respect of same θ and b:

(a) f_i-g_i for each $i\in k$ and h_j for each $j\in I$ are

(ρ_i-b) convex, (σ_i, b) - convex and

(γ_j, b_j) convex respectively such that

$$\sum_{i=1}^{k}\overline{\lambda}_i\left[\rho_i+\overline{V}_j\sigma_i\right]+\sum_{j=1}\overline{\mu}_j V_j\ge 0$$

and for all $x\in\lambda, b\left(x,\overline{x}\right)>0$

(b) $\sum_{i=1}^{k}\overline{x}:$c is (ρ , b) – pseudoconvex and h_j for each $j\in I$ is (γ_j, b) - quasiconvex such that

$$\sum_{j\in I}\overline{\mu}_i\gamma_j+\rho\ge 0$$

(c) $\sum_{i=1}^{k}\overline{\lambda}_i\left(f_i-\overline{V}_i\, g_i\right)$ is (ρ , b) – pseudoconvex and $\sum_{j\in I}\overline{\mu}_j h_j$ is (γ , b) – quasiconvex such that $\rho+\gamma\ge 0$

Proof: (a) For nay feasible x of (FP) and $j\in I\; h_j(x)\le 0=h_j\left(\overline{x}\right)$

Since $h_j; j\in I$ is (γ_j, b_j) - convex with respect to θ at $\overline{x}$, it follows from above that for $i\in I$.

$$0\ge b_j\left(x,\overline{x}\right)\left[h_j(x)-h_j\left(\overline{x}\right)\right]\ge(x-x)^t\nabla h_j\left(\overline{x}\right)+\gamma_j\,||\,\theta\left(x,\overline{x}\right)||^2$$

As $\overline{\mu}_j\ge 0$ for $j\in I.$ we have

$$(x-x)^t\sum_{j\in I}\overline{\mu}_j\nabla h_j\left(\overline{x}\right)+\sum_{j=1}\overline{\mu}_j\gamma_j\,||\,\theta\left(x,\overline{x}\right)||^2\le 0$$

$$\left(x-\bar{x}\right)^t \sum_{i=1}^{k} \bar{\lambda}_i \left[\nabla f_i\left(\bar{x}\right) - \bar{V}_i \nabla g_i\left(\bar{x}\right)\right]$$

$$+\sum_{i=1}^{k} \bar{\lambda}_i \left[\rho_i + \bar{V}_i\, \sigma_i\right] \ \| \theta\left(x,\bar{x}\right) \|^2 \geq 0$$

Now the (ρ_i , b) – convexity of f_i and the $\left(\sigma_i, b\right)$ convexity of - g_i give us

$$\sum_{i=1}^{K} \bar{\lambda}_i b\left(x,\bar{x}\right)\left[f_i\left(x\right) - \bar{V}_i\, g_i\left(x\right)\right] - \sum_{i=1}^{k} \bar{\lambda}_i\, b\left(x,\bar{x}\right)\left[f_i\left(x\right) - \bar{V}_i g_i\left(\bar{x}\right)\right] \geq 0$$

Since $b\left(x,\bar{x}\right) > 0$ we get

$$\sum_{i=1}^{K} \bar{\lambda}_i \left[f_i\left(x\right) - \bar{V}_i\, g_i\left(x\right)\right] \geq \sum_{i=1}^{k} \bar{\lambda}_i \left[f_i\left(\bar{x}\right) - \bar{V}_i g_i\left(\bar{x}\right)\right]$$

This shows that $\bar{x}$ is optimal for $\left(FP\right)^{\lambda}$, since $\bar{\lambda} > 0$ Lemma 7.4.2 shows that $\bar{x}$ is properly efficient for (FP).

(b) For any feasible x of (FP), $h_j\left(x\right) \leq 0 = h_j\left(\bar{x}\right)$ all $j \in I,$ Hence (γ_j , b) – quasiconvexity of h_j implies

$$b\left(x,\bar{x}\right)\nabla h_j\left(\bar{x}\right)\left(x-x\right)^t + \gamma_j \| \theta\left(x,\bar{x}\right) \|^2 \leq 0$$

i.e. $b\left(x,\bar{x}\right)\left(x-x\right)^t \sum_{j\in I} \bar{\mu}_j \nabla h_j\left(\bar{x}\right) + \sum_{j\in I} \bar{\mu}_j\, r_j \| \theta\left(x,\bar{x}\right) \|^2 \leq 0$

$$b\left(x,\vec{x}\right)\sum_{i\in I}^{K} \bar{\lambda}_i \left(\nabla f_i\left(\bar{x}\right) - \bar{V}_i \nabla g_i\left(\bar{x}\right)\right)\left(x-x\right)^t + \rho \| \theta.\bar{x} \|^2 \geq 0$$

Now (ρ , b)–pseudo convexity of $\sum_{i=1}^{K} \bar{\lambda}_i \left(f_i - \bar{V}_i\ g_i\right)$ shows that $\bar{x}$ is optimal for $\left(FP\right)^{\bar{\lambda}/V}$.

Hence by Lemma 7.4.1 $\bar{X}$ is properly efficient for (FP)

(c) As in (b), (γ , b) – quasi convexity of $\sum_{j\in I} \bar{\mu}_i\, h_j$ implies

$$b\left(x,\bar{x}\right)\sum_{j\in I} \bar{\mu}_i \nabla h_j\left(\bar{x}\right)\left(x-x\right)^t + r \| \theta\left(x,\bar{x}\right) \|^2 \leq 0$$

rest of the proof follows on the same lines as that of (b)

Duality Theorems: Weir [91] gave the multioobjective analogue of Jagannathan [45] and Schaible dual for [FP] and proved the duality results under the conditions of convexity. Then after Kaul and Lyail [50] have considered the Schaiable [2006] type dual and proved duality results under the assumptions of strict pseudo convexity. In a similar way Suneja and Gupta [78] proved duality results under the assumptions of $(x\text{-}x)^t$ – Pseudo convexity, $(x\text{-}x)^t$ – strict pseudo convexity and $(x\text{-}x)^t$ – Pseudoconvexity, $(x\text{-}x)^t$ – strict pseudo convexity and $(x\text{-}x)^t$ convexity. We associate the following schaible type [75] dual multi objective maximization problem given by Kaul and Lyallto (FP).

(FD) Max $V=(V_1, V_2 \ldots. V_k)$

Subject to $\sum_{i=1}^{K} \lambda_i \left[\nabla f_i\left(u\right) - v_i \nabla g_i\left(u\right)\right] + \sum_{j=1}^{K} \mu_i \nabla h_j\left(\mu\right) = 0$

$$\sum_{i=1}^{K} \lambda_i \left[f_i\left(u\right) - v_i g_i\left(u\right)\right] \geq 0$$

$$\sum_{i=1}^{K}\mu_j h_j\left(u\right)\geq 0$$

Where $u\in s,\ v_1\lambda\in R_+^K$ and $\mu\in R_+^m$

Now we give various duality theorems between programmes (FP) and (FD) under the assumptions of (ρ , b) – convexity and its generalizations on the objective and constraint functions.

Weak Duality Theorem: Assume that for all feasible x for (FP) and for all feasible $\left(u,\lambda,\mu,v\right)$ for (FD) either of the following holds at $u\,\text{on}\,x$ with respect to the some $\theta\,\text{and}\,b:$

(i) $f_i - g_i$ for each $i\in k$ and h_i for each $j\in M$ are $\left(\sigma_i,b\right)$- convex respectively such that

$$\sum_{i=1}^{K}\lambda_i\left(\rho_i+v_j\sigma_i\right)+\sum_{j=1}^{m}\mu_j\gamma_j\geq 0$$

and for all $x\in x,\ b(x,\bar{x})>0$

(ii) $\sum_{i=1}^{k}\lambda_i\left(f_i-v_i g_i\right)$ is (ρ , b) – pseudo convex and $\sum_{j=1}^{m}\mu_j h_j$ is $(\gamma,b)-$ quasi convex

Such that

$$\rho+r\geq=0$$

Then Q(x) $\leq v$

Proof: (i) From the respective definitions of (ρ_i , b) – convexity $\left(\sigma_i,b\right)$-convexity and $\left(\gamma_j,b\right)$- convexity of $f_i-g_i\,\text{and}\,h_j$ we get

$$b\left(x,u\right)\sum_{i=1}^{k}\lambda_i\left[f_i\left(x\right)-g_i\left(x\right)\right]\geq b\left(x,u\right)\sum_{i=1}^{k}\lambda_i\left[f_i\left(\mu\right)-V_i g_i\left(\mu\right)\right]$$

$$+\sum_{i=1}^{k}\lambda_i\left[\nabla f_i\left(u\right)-v_i\nabla g_i\left(\mu\right)\right]\left(x-u\right)^t$$

$$+\sum_{i=1}^{K}\lambda_i\,\|\,\theta\left(x_,\mu\right)^2\,\|\left(\rho_i+V_i\sigma_i\right)$$

$$\geq-\sum_{j=1}^{m}\mu_i\left[\nabla h_i\left(u\right)\left(x-u\right)^t+\gamma_j\,\|\,\theta\left(x,u\right)\|^2\right]$$

Con using the duality constraints (8.8), (8.9), (8.11) and b(x, u) > 0

$$\sum_{i=1}^{K}\lambda_i\left[f_i\left(x\right)-V_i g_i\left(x\right)\right]\geq 0$$

Now suppose contrary to the result of the theorem hold i.e. $Q(x)\leq V.$ then there exist some $j\in K$ such that $Q_j\left(x\right)<V_j$ and $Q_i\left(x\right)\leq V_i$ for all $i\in k,$ which gives $f_i\left(x\right)-v_j g_j\left(x\right)<0$ for some $j\in K$ and $f_i\left(x\right)-v_i g_i\left(x\right)\leq 0$ for all $i\in k$

Since $\lambda_i>0\sum_{i=1}^{K}\lambda_i\left[f_i\left(x\right)-v_i g_i\left(x\right)\right]<0$

Which contradicts

(ii) Since x is feasible for (FP). U is feasible to (FD) and $\mu_j\geq 0$ for all $j\in M,$ we have

$$\sum_{j=1}^{m}\mu_i h_j(x) \le \sum_{j=1}^{m}\mu_j h_j(u)$$

The (γ , b) quasiconvexit of $\sum_{j=1}^{m}\mu_j h_j$ gives

$$b(x,u)\sum_{j=1}^{m}\mu_j \nabla h_j(u)(x-u)^t + \gamma \| \theta(x,u\|)^2 \le 0$$

on using the duality constraint and (ρ , b) – Pseudo convexity

$$\sum_{i=1}^{k}\lambda_i(f_i - v_i g_i)(x) \ge \sum_{i=1}^{k}\lambda_i(f_i - v_i g_i)(u)$$

i.e. $\sum_{i=1}^{k}\lambda_i(f_i - v_i g_i)(x) \ge 0$ from

The rest of the proof follows on the same lines as that of (i)

(Strong Duality): Assume $\exists$ some $\bar{x} \in S$ such that $h_j(\bar{x}) < 0$ for $j \in M$ and if either of the following holds on S with respect to same θ and b and that either of the following holds on S with respect to same θ and b:

(a) (i) of theorem with $\gamma_j \ge 0$ for each $j \in I$:

(b) $\sum_{i=1}^{k}\lambda_i(f_i - \gamma_i g_i)$ is (ρ , b) – pseudo convex and $\sum_{j=1}^{m}\mu_j h_j$ is (σ, b) quasi convex such that $p + \sigma \ge 0$ and for each $j \in I,\ h_j$ is (γ_j, b)- quasiconvex with $\gamma_j \ge 0$

Then if $\bar{x}$ is properly efficient for (FD) then there exist $\bar{\lambda}, \bar{\mu}$ and $\bar{v}$ (as are given by such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{v})$ is properly efficient for (FD) and the objective values of (FP) and (FD) are equal.

Proof: There exist $\bar{\lambda}_i > 0, i \in k, \bar{\mu}_j \ge 0\, j \in I\ v_i \ge 0, i \in K$ such that it is satisfied (where $\bar{V}_i$ is given as in Lemma setting $\bar{\mu}_i = 0$ for $j \notin I$ gives that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{v})$ is feasible for (FD).

Suppose that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{v})$ is not efficient for (FD) then there exits (u, λ, μ, v) feasible for (FD) and an index j such that $\bar{V}_j < V_j$ and $\bar{V}_i \le V_i$ for all $i \ne j$ for $i \in k$.

Since $Q_i(\bar{x}) = \bar{V}_i$ we get a contradiction to weak duality theorem

Now, suppose that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{v})$ is not properly efficient for (FD). By following exactly on the same lines as that of theorem of Edugo and Hanson we get again a contradiction to the weak duality theorem . Hence $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{v})$ is properly efficient for (FD).

The equality of objective functions (FP) and (FD) follows from $Q_i(\bar{x}) = \bar{V}_i$ for $i \in K$.

(b) The proof follows on the same lines as that of (a) of the theorem.

Converse Duality: Theorem: Let $(\bar{u}, \bar{\lambda}, \bar{\mu}, \bar{v})$ be feasible for (FD) and that there exists a feasible solution $\bar{x}$ for (FP) such that

$$\sum_{i=1}^{k}\bar{\lambda}_i\left[f_i\left(\bar{x}\right)-\bar{V}_i\, g_i\left(\bar{x}\right)\right]=0$$

Then $\bar{x}$ is properly efficient for (FP) is either of the following holds at $\bar{u}$ with respect to x and the same θ and b:

(a) $f_i - g_i$ for $i \in k$ and h_i for each $j \in M$ are (ρ_i, b) convex (σ_i, b) convex and (γ_j, b)- convex respectively such that

$$\sum_{i=1}^{K}\bar{\lambda}_i\left(\rho_i+\bar{V}_i\sigma_i\right)+\sum_{i=1}^{k}\bar{\mu}_j\,\gamma_j\geq 0$$

(b) (ii) of therom

Proof : (a) By following on the similar liens as that of theorem we get

$$\sum_{i=1}^{K}\bar{\lambda}_i\left(f_i\left(\bar{x}\right)-\bar{V}_i\, g_i\left(\bar{x}\right)\right)\geq 0$$

on using and Lemma we conclude that $\bar{x}$ is properly efficient for (FP)

(b) The proof is similar as that of (a) of the theorem.

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