STABILITY ANALYSIS OF A SEIS EPIDEMIC MODEL FOR CONTINUOUS CASE

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Abstract: In this paper, SEIS model with immigration of human population for the spread of carrier dependent infectious disease is proposed and analyzed. Effects of naturals as well as human population density related to environmental factors, which are conductive to the growth of carrier population, are considered. The model is analyzed by the stability of differential equation. Both the disease free and endemic equilibrium are found and their stability is investigated. The analysis of the model shows that whenever carrier population density increases, the spread of an infectious disease is also increases and the disease becomes more endemic due to immigration.

Keywords: Carrier, Environmental discharges, Logistic model, locally asymptotic.

Introduction: Many infectious diseases are spread by direct contact between susceptibles and infectives. Other diseases are spread in the environment and are transmitted to the human population by insects or other vectors [8]. Here we develop and analyze a model for diseases that are transmitted in both ways. This is the case for typhoid fever and other enteric diseases. There are many carrier dependent infectious diseases which afflict human population around the world [5,7,8,9]. However, the third world countries are most affected by such diseases due to lack of sanitation, wide occurrence of carriers such as flies, ticks, mites, etc. which are generally present in the environment. For example, air-borne carriers or bacteria spread diseases such as tuberculosis and measles; while water-borne carriers or bacteria are responsible for the spread of dysentery, gastroenteritis, diarrhea, etc. These carriers transport infectious agents of diseases from infectives to susceptibles and thus spread such diseases in human population. The modeling and analyses of infectious diseases have been conducted by many scientists. In 1976, H.W.Hethcote studied the spread of infectious disease by incorporating the density of carrier population in the model [1]. The size of the carrier population varies and depends on the natural conditions of the environment as well as on various human related factors. The effect of variable carrier population has not been considered in these studies, however the spread of such diseases is very much dependent on the carrier population, the density of which increases due to environmental factors such as temperature, humidity, rain, vegetation, etc. in the habitat[3,9]. The density of carrier population further increases as the human population density increases. With increase in human population density, the effects of human population related factors like discharge of household wastes, open sewage drainage, and industrial effluents in residential areas, open water storage tanks and ponds etc. leads to further growth of carrier population density. This provides a very conducive environment for the growth of these carriers which enhances the chance of carrying more bacteria from infectives to the susceptibles in the population leading to fast spread of carrier dependent infectious diseases. Thus, unhygienic environmental conditions in the habitat caused by human population become responsible for the fast spread of an infectious disease. It is, therefore, reasonable to assume that the carrier population density is governed by a generalized logistic model. The per capita growth rate and the modified carrying capacity of carrier population are taken to be functions of human population density and assumed to increase as the human population density increases [8,9].

Further, it is noted that in most of the epidemic models, the total population size is assumed to be constant as birth and death rate are taken to be equal. But this does not happen if death rate caused by an infectious disease is significantly large, as in the case of cholera, etc. A similar situation also arises where there is no balance between incoming and outgoing population in the region under consideration. Keeping this in view, models with demographic structures have been proposed and analyzed involving variation of total population size, which involve birth rate, death rate, immigration, etc.[2,4,6,s10]

Thus, in this paper, SEIS models for carrier dependent infectious diseases with immigration are proposed and analyzed by considering environmental and human population related factors which are conducive to the growth of carrier population.

This paper is organized as follows: in section 2, basic definitions are reviewed. In section 3, SEIS model with constant immigration. In section 4, we analyze the equilibrium analysis. In section 5, we analyse the stability analysis. Finally we summarise the result in the last section.

Preliminaries:

Definition 2.1: (Equilibrium point): A point x* in the domain of f is said to be an equilibrium point of x(n+1)=f(x_n), if it is a fixed point of f ie.,) f(x*)=x*.

Definition 2.2 (Stable and unstable): The equilibrium point x* of x(n+1)=f(x_n) is stable, if given

ε >o,there exists a δ>o such that $|x_0 - x^*|< \delta$ implies $|f^n(x_0) - x^*|< \varepsilon$ for all n>o.If x* is not stable, then it is called unstable.

Definition 2.3 (Attracting): The point x* is said to be attraction if there exists $|x(0) - x^*|< \eta$ implies $\lim_{n\to\infty} x(n) = x*$.If n $=\infty$, x* is called a global attractor or global attracting.

Definition 2.4 (Asymptotically stable equilibrium point): The point x* is an asymptotically stable equilibrium point if it is stable and attracting. If n = ∞, x* is called a global asymptotically stable.

Defintion 2.5 (Routh-Hurwitz theorem): Consider the characteristic equation $|\lambda I - A| = \lambda^n + b_1\lambda^{n-1} + \cdots + b_{n-1}\lambda + b_n = 0$

Determining the n eigenvalue λ of a real n×n square matrix A, where 1 is the identity matrix. Then the eigenvalue λ all have negative real parts if $\Delta_1 > 0, \Delta_2 > 0 \ldots. \Delta_n > 0$

$$\text{Where } \Delta_k = \begin{vmatrix} b_1 & 1 & 0 & 0\ldots & 0 \\ b_3 & b_2 & b_1 & 1\ldots & 0 \\ b_5 & b_4 & b_3 & b_2\ldots & 0 \\ b_7 & b_6 & b_5 & b_4\ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{2k-1} & b_{2k-2} & b_{2k-3} & b_{2k-4}\ldots & b_k \end{vmatrix}$$

SEIS model with constant immigration: Let H(t) be the total population density which is divided into three sub-classes: the

Susceptible class S(t) and the infective class I(t) and the exposed class E(t). It is assumed that all susceptible are affected by the carrier population of density C(t), which is governed by a general Logistic model. In view of the above and by assuming simple mass action interaction, an SEIS model is proposed as follows:

$$\dot{S} = A - \beta SI - \lambda SC - dS + \gamma I$$
$$\dot{E} = \beta SI + \lambda SC - (d + \epsilon)E$$
$$\dot{I} = \epsilon E - (d + \gamma + \alpha)I \qquad (3.1)$$
$$\dot{H} = A - dH - \alpha I$$
$$\dot{C} = s(H)C - \frac{s_0 C^2}{L(H)} - s_1 C$$

which together H=S+E+I

S(o)=$S_0 > 0, I(0) = I_0 \geq 0,$

$E(0) = E_0 \geq 0, H(0) = H_0 > 0,$

C(o) =$C_0 \geq 0$

In model (3.1), A is a constant immigration rate of human population from outside the region under consideration, d is natural death rate constant, β and λ are the transmission coefficients due to infective and carrier population respectively, α is the disease related death rate constant, γ is the recovery rate constant and $1/_\epsilon$ is the mean latent and exposed period. The constant s_1 is the death rate coefficient of carriers due to natural factors as well as by control measures. We may note here that if the growth rate and death rate of carrier population are balanced, then it may tend to zero.

In (3.1), s(H) is the growth rate per capita of the carrier population density such that s(H) -s_1 is its intrinsic growth rate as compared to the usual logistic model. In view of the assumption that the growth rate per capita increases as the human population density increases, we have,

$$s(0) = s_0 > 0 \text{ and } s'(H) \geq 0 \qquad (3.2)$$

where s_0 is the value of s(H) when H = 0.It may be pointed out here that, when s(H) is independent of H.It takes constant value and that value assumed to be s_0.

Similarly, L(H) is the modified carrying capacity of the carrier population and its value as compared to the usual logistic model is: L(H)$\left[\frac{s(N)-s_1}{s_0}\right]$.We assume here that this modified carrying capacity increases with human population density, so we have,

$$L(o) = L_0 > 0 \text{ and } L'(H) \geq 0 \qquad (3.3)$$

Where L_0 is the value of L(H) when H = 0. It may be pointed out here that, when L(H) is independent of H. It takes constant value and that value assumed to be L_0.

From the last equation of (3.1), (3.2) and (3.3), it is noted that even in absence of human population related factors, the carrier population density increases in its natural environment and it tends to $L_0\left[1 - \frac{s_1}{s_0}\right]$which may become zero if $s_1 \to s_0$.

Equilibrium Analysis: To analyze the model (3.1), we consider the following reduced system (since S+E+I= H).

$$\dot{E} = \beta(H - E - I)I + \lambda(H - E - I)C - (d + \epsilon)E$$
$$\dot{I} = \epsilon E - (d + \gamma + \alpha)I \qquad (4.1)$$
$$\dot{N} = A - dH - \alpha I$$
$$\dot{C} = s(H)C - \frac{s_0 C^2}{L(H)} - s_1 C$$

The results of equilibrium analysis are give in the following theorems.

Theorem 4.1: There exist following three equilibria of the system (3.1),

1. P_0 =(0, 0,A/d, 0)
2. $P_1 = (\bar{E}, \bar{I}, \bar{H}, 0)$ which exists if $s(\bar{H}) - s_1 > 0$ where

$$\bar{E} = \frac{[\beta A\epsilon - d(d + \epsilon)(d + \gamma + \alpha)]]d + \gamma + \alpha]}{\beta\epsilon[\alpha\epsilon + d\epsilon + d(d + \gamma + \epsilon)]}$$
$$\bar{H} = \frac{A\beta(d\epsilon + d + \gamma + \alpha) + \alpha d(\alpha + d + \gamma)(d + \epsilon)}{d\beta\big(\alpha\epsilon + d\epsilon + d + \epsilon(d + \gamma + \alpha)\big)}$$

And

$$\bar{I} = \frac{\beta A\epsilon - d(d + \epsilon)(d + \gamma + \alpha)}{\beta\big(\alpha\epsilon + d\epsilon + d(d + \alpha + \gamma)\big)}$$

3. $P_2 = (E^*, I^*, H^*, C^*)$

Proof: The existence of $P_0\, and\, P_1$ is obvious. The nontrivial equilibrium point $P_2 = (E^*, I^*, H^*, C^*)$ is given by the solution of the following set of equations,

$$\beta(H-E-I)I+\lambda(H-E-I)C-(d+\epsilon)=0 \quad (3.5)$$

$$I=\frac{A-dN}{\alpha} \quad (3.6)$$

$$E=\frac{(d+\gamma+\alpha)(A-dH)}{\alpha\epsilon} \quad (3.7)$$

$$C=\frac{[s(H)-s_1]L(H)}{s_0} \quad (3.8)$$

Now putting the value of I, E, C in equation (3.5) then whole equation reduces to H. So we can write,

$$F(H)=\beta\left[H-\frac{(d+\gamma+\alpha)(A-dH)}{\alpha\epsilon}-\frac{A-dH}{\alpha}\right]\left[\frac{A-dH}{\alpha}\right]+\lambda\left[H-\frac{(d+\gamma+\alpha)(A-dH)}{\alpha\epsilon}-\frac{A-dH}{\alpha}\right]\frac{[s(H)-s_1]L(H)}{s_0}-(d+\epsilon)\left[\frac{(d+\gamma+\alpha)(A-dH)}{\alpha\epsilon}\right] \quad (3.9)$$

It is easy to observe that F(0) < 0 and F(A/d) > 0.Therefore, there exists a root $H^*$of F(H) = 0 in 0<H<A/d. Hence, there exists a unique root N^*given by $F(H^*)=0$. Knowing the value of H^*,the values of $I^*, E^* and\ C^*$ can be computed from equations (3.6),(3.7) and (3.8) as $H^*<\frac{A}{d}$.

It may be noted from (3.8) that $\frac{dC}{dH}>0$ in view of (3.2) and (3.3). Hence C increases as H increases. Also, from (3.5) we get $\frac{dI}{dC}>0$. Therefore, it is clear that the density of infective population increases as the density of carrier population increases.

Stability Analysis: Now, we study stability of equilibria P_0 , P_1 and P_2.the local stability results of these equilibria are stated in the following theorem.

Theorem: 5.1: The disease free equilibrium P_0 is unstable whenever s(A/d) >s_1.Again if P_0 is stable if s(A/d) <s_1.

Proof: The variational matrix M_0 for the system (4.1) corresponding to the equilibrium P_0 = (0, 0, A/d, 0) is given by

$$M_0=\begin{pmatrix}-(d+\epsilon) & \beta(A/d) & 0 & \lambda(A/d)\\ \epsilon & -(d+\gamma+\alpha) & 0 & 0\\ 0 & -\alpha & -d & 0\\ 0 & 0 & 0 & s(A/d)-s_1\end{pmatrix}$$

The Eigenvalues of M_0 are given

$$\mu_1=-(d+\epsilon), \mu_2=-(d+\gamma+\alpha),$$

$\mu_3=-d, and\ \mu_4$ =s(A/d) - s_1.

Since one eigenvalue s(A/d) >s_1 is positive.Therefore,P_0 is unstable. If s(A/d) <s_1,then P_0 is stable.

Theorem: 5.2: The carrier free equilibrium P_1 is unstable if s(A/d) >s_1.

Proof: The variational matrix M_1 for the system (4.1) corresponding to the equilibrium $P_1=(\bar{E},\bar{I},\bar{H},0)$ is given by

$$M_1=\begin{pmatrix}-\beta\bar{I}-(d+\epsilon) & \beta(\bar{H}-\bar{E})-2\beta\bar{I} & \beta\bar{I} & \lambda(\bar{H}-\bar{E}-\bar{I})\\ \epsilon & -(d+\gamma+\alpha) & 0 & 0\\ 0 & -\alpha & -d & 0\\ 0 & 0 & 0 & s(\bar{H})-s_1\end{pmatrix}$$

Here $\mu_1=-\beta\bar{I}-(d+\epsilon)$, $\mu_2=-(d+\gamma+\alpha)$, $\mu_3=-d, and\ \ \mu_4$ =s(A/d) - s_1.

Since given that s(A/d)-$s_1>0$. Therefore, one of the eigenvalue is positive. Therefore, carrier free equilibrium is unstable.

Theorem: 5.3: The endemic equilibrium P_2 is locally asymptotically stable provided

$$a_0a_1a_2>a_2{}^2+a_0{}^2a_3$$

where,

$$a_0=(\beta I^*+\lambda C^*+(d+\epsilon)+2d+\gamma+\alpha)-(s(H^*)-s_1)$$

$$a_1=(\beta I^*+\lambda C^*+(d+\epsilon))(2d+\gamma+\alpha)-(s(H^*)-s_1)(\beta I^*+\lambda C^*+(d+\epsilon)+2d+\gamma+\alpha)+d(d+\gamma+\alpha)+2\beta I^*+\lambda C^*-\beta(H^*-E^*)$$

$$a_2=d(\beta I^*+\lambda C^*+(d+\epsilon))(d+\gamma+\alpha)-d[(d+\gamma+\alpha)(s(H^*)-s_1)+\epsilon(\beta(H^*-E^*)-2\beta I^*-\lambda C^*)]-\epsilon d[\beta(H^*-E^*)-2\beta I^*-\lambda C^*]+\alpha\epsilon(\beta I^*+\lambda C^*)$$

$$a_3=d(s(H^*)-s_1)[\epsilon(\beta(H^*-E^*)-2\beta I^*-\lambda C^*)-(\beta I^*+\lambda C^*+(d+\epsilon)(d+\gamma+\alpha)]++\alpha\epsilon\lambda(H^*-E^*-I^*)(s'(H^*)C^*+\frac{s_0C^{*2}}{[L(H^*)]^2})$$

Remark: It can be seen that the condition (4.1) is automatically satisfied when s(H) and L(H) are independent of H, i.e., when the density of carrier population remains unaffected due to human population related factors.

It may pointed out here that a region of attraction for the system (3.1) can be given by

$$\Omega=\{(E,I,H,C): 0\le I\le E\le H\le\frac{A}{d}, 0\le\ C\le C_m\}$$

$$C_m=\frac{[s(A/d)-s_1]L(A/d)}{s_0}$$

which attracts all solutions initiating in the positive quadrant.

Conclusion: In this paper, we have proposed and analyzed SEIS models with immigration for carrier dependent infectious diseases by considering that the density of carrier population increases in the habitat by environmental and human population related factors. It has been assumed that the density of the carrier population is governed by a general logistic model, the growth rate per capita and the modified carrying capacity of which increase as the human population density increases. These models have been analyzed by using stability theory of differential equations. It has been shown that, the number of infectives increases as the density of carrier population increases due to environmental and human population related factors leading to fast spread of such infectious diseases. Futher, it has also been shown that these infectious diseases become more endemic due to immigration.

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