SOME RESULTS ON APPROXIMATIONS OF PRIME s-IDEALS IN SEMINEARRINGS

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Abstract: We consider a seminearring S. We introduce the notions of lower and upper approximations for the ideals of seminearrings. We study the properties of prime ideals in seminearrings using congruence relations and complete congruence relations. In particular we obtain characterizations of prime ideals and complete prime ideals in the form of lower and upper approximations.

Introduction: The algebraic systems with binary operations of addition and multiplication satisfying all the ring axioms except possibly one of the distributive laws and commutativity of addition are called Nearrings. A semiring is an algebraic system which is closed and associative under two operations, usual addition, multiplication, and satisfies both distributive laws. A generalization of both the concepts nearring and semiring, is seminearring S, which is an algebraic system with two binary operations usual addition and usual multiplication such that S forms a semigroup with respect to both the operations, and satisfies the right distributive law. A right semi nearring S is said to have an absorbing zero 0 if a + 0 = 0 + a = a and a. 0 = 0. a = 0 for all a ∈ S.A natural example of a Semi nearring is obtained by considering the operations usual addition and composition of mappings on a set of all mappings of an additive semigroup S into itself.

We denote S for a seminearring with absorbing zero throughout this paper.

For preliminary definitions and results on seminearrings,we refer Golan [1],Javed Ahsan [2],[3], Pilz [4], and Weinert H.J. [5].

A subset I of a seminearring S is a right (respectively, left) s-ideal

if (i) x+y ∈ I,

(ii) xr ∈ I (right s-ideal), rx∈ I (left s-ideal) for all x, y∈ I and r ∈ S.

A left (or right) s-ideal I of S is called a left (or right) sk-ideal of S if y∈ I and x ∈ S, x + y ∈ I implies that x ∈ I.

An s-ideal P in a seminearring S is said to be prime if for any two s -ideals A, B in S such that AB ⊆ P implies A ⊆ P or B ⊆ P.

For any two subsets of A and B of S, we write AB = {xy / x∈A, y ∈ B}.

Let θ be an equivalence relation on S. θ is said to be a congruence relation on S if

(a, b), (c,d) $\in \theta$ implies that (a + c, b + d) $\in \theta$ and (ac, bd) $\in \theta$.

A partition of S (assume that S is non-empty) is a family $\wp$ of non-empty subsets of S such that each element of S is contained in exactly one element of $\wp$. Each partition $\wp$ induces an equivalence relation θ on S by (a, b) $\in \theta$ if and only if a and b are in the same class of $\wp$.

Also it is clear that, each equivalence class induces a partition of N whose classes have the form $[x]_\theta = \{y \in S / (x, y) \in \theta\}$ which denotes the equivalence class of θ determined by x.

The pair (S, θ) where θ is an equivalence relation on S, is called an approximation space.

Lemma: Let θ be a congruence relation on S. If a, b ∈ S, then

$i) [a]_\theta + [b]_\theta \subseteq [a+b]_\theta$

$ii) [a]_\theta . [b]_\theta \subseteq [a.b]_\theta$

Proof:

1. Take x ∈ S. Suppose x $\in [a]_\theta + [b]_\theta$.

   Then there exists y, z ∈ S such that x = y + z and y $\in [a]_\theta$,z $\in [b]_\theta$.

   This means (a, y) , (b, z) $\in \theta$.

   Therefore (a + b,y + z) = (a + b,x) $\in \theta$. Hence x$\in [a+b]_\theta$

   Thus $[a]_\theta + [b]_\theta \subseteq [a+b]_\theta$.

2. Take p $\in [a]_\theta . [b]_\theta$ Then p = r. s for some r $\in [a]_\theta$ and s $\in [b]_\theta$.

   This implies that (a,x) $\in \theta$ and (b,y) $\in \theta$.

Since θ is a congruence relation, we get (ab, xy) $\in \theta$.

Therefore $[a]_\theta . [b]_\theta \subseteq [ab]_\theta$.

Rough s-ideals in seminearrings:

Definition: Let (S,θ) be an approximation space and let θ be a congruence relation on S. For a non-empty subset A of S, the sets $\theta_-(A)$ = {x $\in$ S/ $[x]_\theta \subseteq A$ } and $\theta^-(A)$ = {x $\in$ S / $[x]_\theta \cap A \neq \phi$} are called the θ -lower and θ-upper approximations of A.

θ(A) is called the rough set with respect to θ if θ_-(A) $\neq \theta^-(A)$

It is clear that $\theta^-(S) = S$.

Definition: A congruence θ on S is called complete if $[a]_\theta . [b]_\theta = [ab]_\theta$ for all a, b $\in$ S.

For any s-ideal K of S and a, b $\in$ S, we say that a is congruent to b mod Kwritten as a $\equiv$ b(modK) if a + b $\in$ K.

We consider the approximation space as (S,K).

For any subset X of S, we use X^c to denote S -X.

The proof of the following Lemma and the Proposition are straight forward verification.

Lemma: For every approximation space (S,K) and subsets I, J of S, the following hold.

(1) $\theta_{-K}(S-I) = S - \theta_K^-(I)$

(2) $\theta_K^-(S-I) = S - \theta_{-K}(I)$

(3) $\theta_K^-(J) = \left(\theta_K^-(J^c)\right)^c$

(4) $\theta_{-K}(J) = \left(\theta_K^-(J^c)\right)^c$

Proposition: Let θ and λ be congruence relations on a seminearring S.

If A and B are nonempty subsets of S, then the following hold.

1. $\theta_-(A) \subseteq A \subseteq \theta^-(A)$
2. $\theta^-(A \cup B) = \theta^-(A) \cup \theta^-(B)$
3. $\theta_-(A \cap B) = \theta_-(A) \cap \theta_-(B)$
4. $A \subseteq B \Rightarrow \theta_-(A) \subseteq \theta_-(B)$
5. $A \subseteq B \Rightarrow \theta^-(A) \subseteq \theta^-(B)$
6. $\theta_-(A \cup B) \supseteq \theta_-(A) \cup \theta_-(B)$
7. $\theta^-(A \cap B) \subseteq \theta^-(A) \cap \theta^-(B)$
8. $\theta \subseteq \lambda \Rightarrow \theta_-(A) \supseteq \lambda_-(A)$
9. $\theta \subseteq \lambda \Rightarrow \theta^-(A) \supseteq \lambda^-(A)$

Remark: Consider the seminearring S =Z_{15} and an s-ideal I = {0, 5, 10} with subsets A = {1,2, 3, 6, 7, 9, 11, 13} and B = {1, 6, 7, 8, 10, 11} of S. Then it is clear that conditions (in general), $\theta_-(A \cup B) \varnothing \theta_-(A) \cup \theta_-(B)$ and $\theta^-(A) \cap \theta^-(B) \varnothing \theta^-(A \cap B)$.

Definition: Let θ be a congruence relation on S. Then θ is said to be complete

if $[a]_\theta + [b]_\theta = [a+b]_\theta$, for any a , b $\in$ S.

Proposition: Let θ be a congruence relation on S. If A and B are non empty subsets of S, then

(i) $\theta^-(A).\theta^-(B) \subseteq \theta^-(AB)$.

(ii) $\theta_-(A).\theta_-(B) \subseteq \theta_-(AB)$

Rough prime s-ideals in seminearrings:

In this section we prove that a prime s - ideal of a seminearring S is invariant under the upper and lower approximations.

Theorem: Let θ be a congruence relation of S and P a prime s- ideal of S such that $\theta^-(P) \neq S$, then P is an upper rough prime s- ideal of S.

Proof: First we show that $\theta^-(P)$ is a s-ideal of S.

Let a, b $\in \theta^-(P)$.

Then $[a]_\theta \cap P \neq \phi$ and $[b]_\theta \cap P \neq \phi$. So there exist x $\in [a]_\theta \cap P$ and y $\in [b]_\theta \cap P$.

Since P is an ideal of S, we have that x + y $\in$ P.

Now x + y $\in [a]_\theta + [b]_\theta = [a+b]_\theta$ (by Remark 2.5).

Therefore $[a+b]_\theta \cap P \neq \phi$. And so a + b $\in \theta^-(P)$.

Take p $\in \theta^-(P)$ and s $\in$ S.

Then there exists x $\in [p]_\theta \cap P$ and (x; p) $\in \theta$.

Since θ is a congruence relation on S, We get that (xs, ps) $\in$ P.

This implies that xs $\in [ps]_\theta$.

Therefore xs $\in [ps]_\theta \cap P$ and hence ps $\in \theta^-(P)$.

In a similar way we can show that sp $\in \theta^-(P)$.

Therefore $\theta^-(P)$ is a s-ideal of S.

Now we prove $\theta^-(P)$ is a prime s-ideal.

On contrary, suppose that $IJ \subseteq \theta^-(P)$ where I, J are s-ideals of S and

$I \not\subseteq \theta^-(P)$ And J $\not\subseteq \theta^-(P)$.

Then there exists x $\in$ I such that x $\notin \theta^-(P)$ and there exist y $\in$ J such that y $\notin \theta^-(P)$.

This means that $[x]_\theta \cap P = \phi = [y]_\theta \cap P$. This implies x,y $\notin P$.

Since P is prime s-ideal,we have xy $\notin P$, a contradiction.

Hence P is an upper rough prime s- ideal of S.

Theorem: Let S be a seminearring in whch every element has additive inverse and θ a congruence relation of S. If P is a prime s - ideal of S such that $\theta_-(P) \neq \phi$ then P is a lower rough prime s -ideal of S.

Proof: First we prove that $\theta_-(P) \neq \phi$, then $\theta_-(P)$ = P.

Since $\theta_-(P) \neq \phi$, there exists x $\in$ S such that x $\in$ $\theta_-(P)$. Now x$\in [x]_\theta \subseteq P$,

We have $\theta_-(P) \subseteq$ P.

Suppose a $\in$ P. Now

$$[0]_\theta = [x+(-x)]_\theta = [x]_\theta + [-x]_\theta$$
$$= [x]_\theta + ([-x]_\theta) \subseteq P + P \subseteq P$$

.

Since P is an ideal of S, we have that $a + [0]_\theta \subseteq a + I \subseteq I$.

Also

$$x \in a + [0]_\theta \Leftrightarrow x - a \in [0]_\theta \Leftrightarrow (x,a) \in \theta \Leftrightarrow x \in [a]_\theta \subseteq P$$

.

Since $\theta_-(P) = P$ we have $\theta_-(P)$ is a prime ideal of S.

Conclusions: Seminearrings are generalized algebraic structures of the concepts namely, semirings, nearrings. seminearrings have shown to be useful tool in studying automata and formal languages(cf.[1], [5] and [6]). A preliminary algebraic approach to investigate basic lower and upper approximations of substructures like s-ideals and prime s-ideals have been discussed.

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