SOME CHARACTERISATIONS OF 2-FUZZY N-N-B- INNER PRODUCT SPACE

Thangaraj Beaula
Dept of Mathematic/T.B.M.L College/Porayar – 609307/India
Daniel Evans
Dept of Mathematics/T.B.M.L College/Porayar – 609307/India

Abstract: The purpose of this paper is to introduce the new notion of 2-fuzzy n-n b-semi inner product space and 2-fuzzy n-n b-Hilbert space .Some characterizations of this space is dealt with.

Keywords: 2-Fuzzy N-N Inner Product Space, 2-Fuzzy N-N B-Hilbert Space, Bessel's Inequality, Riesz Theorem.

Introduction: The concept of fuzzy sets was introduced by Zadeh [22] in 1965 which began a new revolutionary field in mathematics. The theory of 2-norm on a linear space was given by Gahler[9](1964). In 1984 Katsaras [12] gave the notion of fuzzy norm on a linear space. Several different definitions of fuzzy normed spaces were given by Cheng and Mordeson[2], Bag and Samanta[1]. R.M.Somasundaram and ThangarajBeaula[17] defined the notion of fuzzy 2-normed linear space, {F(X),N}, further improvised by proving some standard results in [21]. The concept of 2-inner product space was introduced by C.R.Diminnie, S.Gahler and A.White [4]. Further various researchers established new notions of fuzzy normed linear space[6,7,8,12,15,16] and fuzzy inner product space in [5,12,13] . Vijayabalaji and Thillaigovindan [18] introduced fuzzy n-inner product space as a generalization of the concept of n-inner product space given by Y.J.Cho, M.Matic and J.Pecaric in [3]. ThangarajBeaula and Daniel Evans extended the notion of [18] to 2 fuzzy n-n inner product space in [20] P. K. Harikrishnan, P. Riyas and K. T. Ravindran gave the proof of Riesz theorem for 2- inner product spaces which hold for b-linear functional. The notion of 2- fuzzy n - b metric space was given by ThangarajBeaula and ChristinalGunaseeli[19]

In this paper the notions of 2-fuzzy n-n b-inner product space and 2-fuzzy n-n b-Hilbert space are introduced and some standard results are proved.

Preliminaries:
Definition2.1: ([18]):
Let n ∈N and X be a real linear space of dimension greater than or equal to n. A real valued function $\|.,\dots,.\|$ on X×…×X (n-times) = X^n satisfying the following four properties
i) $\|x_1,\dots,x_n\|$= 0 if and only if $x_1,\dots,x_n$ linearly dependent.
ii) $\|x_1,\dots,x_n\|$ is invariant under any permutation
iii) $\|x_1,\dots,\alpha x_n\|=|\alpha|\|x_1,\dots,x_n\|$, for any α is real
iv) $\|x_1,\dots,x_{n-1},y+z\| \leq \|x_1,\dots,x_{n-1},y\|+\|x_1,\dots,x_{n-1},z\|$
is called an n-norm on X and the pair (X, $\|.,\dots,.\|$) is called n-normed linear space.

Definition 2.2: ([22]): A fuzzy set in X is a map from X to [0,1], it is an element of $[0,1]^X$

Definition 2.3: ([17]): Let X be a nonempty and F(X) be the set of all fuzzy sets in X.
If f ∈F(X) , f ={(x,μ)|x∈X and μ∈(0,1]}then f is a bounded function for f(x) ∈ [0,1]
((i.e) |f(x)|≤1). Let K be the space of real numbers, F(X) is a linear space over the field K where the addition and scalar multiplication are defined by
f+g ={(x,μ)+(y,η)}={(x+y,μ∧η)|(x,μ)∈f and (y,η)∈g}
and kf = {(kf,μ)|(x,μ)∈f,} where k∈K.

The linear space F(X) is said to be normed space if for every f ∈F(X), there is associated a non-negative real numbers ||f|| called the norm of f in such a way that

(i) $||f||= 0$ if and only if $f = 0$ For $||f||=0 \Leftrightarrow \{||(x,\mu)|||(x,\mu)\in f\}= 0$, $\Leftrightarrow x = 0$, $\mu\in(0,1] \Leftrightarrow f =0$.
(ii) $||kf||=|k|||f||$, $k\in K$.
For $||kf||=\{||k(x,\mu)||/(x,\mu)\in f$
and $k\in K\}=\{|k|||(x,\mu)||/(x,\mu)\in f \}=|k|||f||$.
(iii) $||f +g||\leq||f||+||g||$ for every $f,g\in F(X)$.
For $||f+g||=\{||(x,\mu)+(y,\eta)||/x,y\in X,\mu,\eta\in(0,1]\}$
$=\{||(x+y),(\mu\wedge\eta)|| /x,y\in X,\mu,\eta\in(0,1]\}$
$\leq\{||(x,\mu\wedge\eta)||+||(y,\mu\wedge\eta)||/(x,\mu)\in f$ and $(y,\eta)\in g\}=||f||+||g||$.

Definition 2.4: ([5]): Let $F(X^n)$ be a linear space over a real field. A fuzzy subset N of $F(X^n)\,^n\times R$ is called 2-fuzzy n-n norm if and only if
(N_1) for all $t \in R$, $t \leq 0$, $N (f_1,\ldots,f_n, t) = 0$
(N_2) for all $t \in R$, $t > 0$, $N (f_1,\ldots,f_n, t) = 1$ if and only if $f_1,\ldots,f_n$ are linearly dependent
(N_3) $N (f_1,\ldots,f_n, t)$ is invariant under any permutation of $f_1,\ldots,f_n$
(N_4) for all $t \in R$, $t > 0$,
$$N (f_1,\ldots,cf_n, t) = N(f_1,\ldots,f_n, \frac{t}{|c|})$$
(N_5) for all $s,t\in R$, $N (f_1,\ldots,f_n + f_n', s+t) \geq$
$\min \{N(f_1,\ldots,f_n,s),N(f_1,\ldots,f_n',t)\}$
(N_6) $N (f_1,\ldots.f_n,t)$ is a non-decreasing function of $t\in R$ and $\lim_{t\to\infty} N(f1, \ldots, fn, t)$
The $(F(X^n)^n, N)$ is called a 2- fuzzy n-n normed linear space.

Definition 2.5: ([21]): Let F(X) be a linear space over the complex field C. The fuzzy subset η defined as a mapping from $F(X)\times F(X)\times C$ to [0,1] such that for all $f,g,h\in F(X)$ and $\alpha \in C$
(I_1) for $s,t\in C,\eta(f+g,h,|t| + |s|)$
$\geq \min \{\eta(f,h,|t|),\eta(g,h,|s|)\}$
(I_2) for $s,t\in C,\eta(f,g, |st|)$
$\geq \min \{\eta(f,f,|s|^2), \eta(g,g,|t|^2) \}$
(I_3) for $t \in C$,$\eta(f,g,t) = \eta(g,f,t)$
(I_4) $\eta(\alpha f,g,t) = \eta(f,g,\frac{t}{\alpha})$, $\alpha\neq 0$
(I_5) $\eta(f,f,t) = 0$ for all $t \in C\backslash R^+$
(I_6) $\eta(f,f,t) = 1$ for all $t > 0$ if and only if $f = 0$
(I_7) $\eta(f,f,.) : R \to I$ (=[0,1]) is a monotonic
non-decreasing function of R and
$\lim \eta(f,f,t) = 1$ as $t\to \infty$
Then η is said to be a 2-fuzzy inner product space on F(X) and the pair $(F(X),\eta)$ is called a 2-fuzzy inner product space.

Definition 2.6: ([20]): Let $F(X^n)$ be a linear space over a real field. A fuzzy subset N of $F(X^n)\,^n\times R$ is called 2-fuzzy n-n norm if and only if
(N_1) for all $t \in R$, $t \leq 0$, $N (f_1,\ldots,f_n, t) = 0$
(N_2) for all $t \in R$, $t > 0$, $N (f_1,\ldots,f_n, t) = 1$ if and only if $f_1,\ldots,f_n$ are linearly dependent
(N_3) $N (f_1,\ldots,f_n, t)$ is invariant under any permutation of $f_1,\ldots,f_n$
(N_4) for all $t \in R$, $t > 0$,
$$N (f_1,\ldots,cf_n, t) = N(f_1,\ldots,f_n, \frac{t}{|c|})$$
(N_5) for all $s,t\in R$, $N (f_1,\ldots,f_n + f_n', s+t)$
$\geq \min \{N(f_1,\ldots,f_n,s),N(f_1,\ldots, f_n',t)\}$
(N_6) $N (f_1,\ldots.f_n,t)$ is a non-decreasing function of $t\in R$ and $\lim_{t\to\infty} N(f_1, \ldots, f_n, t)=1$
The space $(F(X^n)^n, N)$ is called a 2- fuzzy n-n normed linear space.

Definition 2.7: ([20]): Let $F(X^n)$,be a linear space over $\mathbb{C}$. Define a fuzzy subset η defined as a mapping from $[F(X^n)]^{n+1}\times \mathbb{C}$ to [0,1] such that
$(f_1,....f_n, f_{n+1})\in [F(X^n)]^{n+1}\alpha\in \mathbb{C}$ satisfying the following conditions.
(I_1) for $g,h\in F(X)$, $s,t \in s$
$\eta (f_1+ g, h, f_2,...,f_n, |t| + |s|) \geq$
$\min \{\eta (f_1,h, f_2,...,f_n, |t|), \eta (g, h, f_2,...,f_n, |s|)\}$
(I_2) for $s,t\in\mathbb{C}$ $\eta (f_1,g, f_2,...,f_n, |st|) \geq$
$\min \{\eta (f_1, f_1, f_2,...,f_n, |s|^2, \eta (g, g, f_2,...,f_n, |t|^2\}$
(I_3) for $t\in \mathbb{C}$
$\eta (f_1, g, f_2,...,f_n,|t|) = \eta (g, f_1, f_2,...,f_n,|t|)$
(I_4) $\alpha_1,\alpha_2, \in \mathbb{C}$, $\alpha_1\neq 0, \alpha_2\neq 0$
$\eta (\alpha_1 f_1, \alpha_2 f_1, f_2,...,f_n,t), = \eta (f_1, f_1, f_2,...,f_n, \frac{t}{|\alpha_1 , \alpha_2|})$
(I_5) $\eta(f_1,f_1, f_2,...,f_n,t), = 0$ $\forall t \in\mathbb{C} / R^+$
$\eta (f_1,f_1, f_2,...,f_n,t) = 1\ \forall t > 0$ if and only if $f_1,....f_n$ are linearly dependent.
(I_6) $\eta(f_1,g, f_2,...,f_n, t)$ is invariant under any permutation of $f_1,g, f_2,...,f_n$
(I_7) $\forall$ $t >0$,$\eta(f_1, f_1, f_2,...,f_n, t) =\eta(f_2, f_2, f_1, f_3,...,f_n, t)$
$(I_8)\eta(f_1,g,f_2,...,f_n, t)$ is a monotonic non-decreasing function of $\mathbb{C}$ and $\lim \eta (f_1, g, f_2, \dots , f_n, t) = 1$
Then η is said to be the 2- fuzzy n-n inner product $F(X)^n$ and the pain $(F(X)^n ,\eta)$ is called
2 - fuzzy n-n IPS.

Definition 2.8: ([20]): Let $(F(X^n),\eta)$ be a 2-fuzzy n-n IPS satisfying the condition $\eta (f_1, f_1 f_2,...,f_n, t^2) > 0$, when $t > 0$ implies that $f_1 f_2,...,f_n$ are linearly dependent. Then for all $\alpha\in(0,1)$, define $\|f_1,...,f_n\|_\alpha = \inf \{t ; \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha \}$ a crisp norm on $F(X^n)$ called the α-n-n norm on $F(X^n)$ generated by η.

Fuzzy N-N B-Semi Inner Product Space:
Definition 3.1: Let $(F(X^n),\|., \dots , .\|)$ be a n-normed space and $f,g\in F(X^n)$, then f is said to be b-orthogonal to g if and only if there exists $d\in F(X^n)$ such that for every α,
$\|f, d \dots , f_n\|_\alpha \neq 0$, $\|f + \alpha g, d \dots , f_n\|_\alpha \leq \|f, d \dots , f_n\|_\alpha$ and $g\neq 0$

Definition 3.2: Let $F(X^n)$ be a linear space over a real field. A fuzzy subset N of $[F(X^n)]^n\times R$ is called fuzzy n-n b-norm if and only if
(N_1) for all $t \in R$, $t \leq 0$, $N (f_1,...,f_n, t) = 0$
(N_2) for all $t \in R$, $t > 0$, $N (f_1,...,f_n, t) = 1$ if $f_1,...,f_n$ are linearly dependent
(N_3) $N (f_1,...,f_n, t)$ is invariant under any permutation of $f_1,...,f_n$
(N_4) for all $t \in R$, $t > 0$,
$N (f_1,...,cf_n, t) = N(f_1,...,f_n, \frac{t}{|c|})$
(N_5) for all $s,t\in R$, $N (f_1,...,f_n + f_n^{'}, s+t) \geq$
$u\{\min \{N(f_1,...,f_n,s),N(f_1,..., f_n^{'},t)\}$, $0 \leq u \leq 1$
(N_6) $N (f_1,....,t)$ is a non-decreasing function of $t\in R$ and $\lim_{t\to\infty} N(f_1, \dots , f_n, t)$
The space $(F(X^n)^n, N)$ is called a
2- fuzzy n-n b- semi normed linear space.

Definition 3.3: Let $F(X^n)$,be a linear space over R. Define a fuzzy subset η' as a mapping from $[F(X^n)]^{n+1}\times R$ to [0,1] such that $(f_1,....f_n, f_{n+1})\in [F(X^n)]^{n+1}$ and $\alpha \in R$ satisfying the following conditions.

(I_1) for g,h∈ F(X), s,t ∈ R
$\eta'(f_1+g, h, f_2,\ldots,f_n, t+s)$
$\geq u[\min\{\eta'(f_1,h,f_2,\ldots,f_n, t), \eta'(g, h, f_2,\ldots,f_n, s)\}]$
(I_2) for s,t∈R $\{\eta'(f_1,g, f_2,\ldots,f_n,\sqrt{st})$
$\geq u^2[\min\{\eta'(f_1, f_1, f_2,\ldots,f_n, s),\eta'(g, g, f_2,\ldots,f_n, t\}]$
(I_3) for t∈ R
$\eta'(f_1, g, f_2,\ldots,f_n,t) = \eta'(g, f_1, f_2,\ldots,f_n,t)$
(I_4) $\alpha_1,\alpha_2, \in R$, $\alpha_1\neq 0, \alpha_2\neq 0$

$$\eta'(\alpha_1 f_1, \alpha_2 f_1, f_2,\ldots,f_n,t), = \eta'(f_1, f_1, f_2,\ldots,f_n, \frac{t}{|\alpha_1,\alpha_2|})$$

(I_5) for all t∈R with $t \leq 0$, $\eta'(f_1,f_1, f_2,\ldots,f_n,t) = 0$
(I_6) for all t∈R with $t > 0$ $\eta'(f_1,f_1, f_2,\ldots,f_n,t) = 1$ if
$f_1,\ldots.f_n$ are linearly dependent.
(I_7) $\eta'(f_1,g, f_2,\ldots,f_n, t)$ is invariant under any permutation of $(f_2,\ldots,f_n)$
(I_8) $\forall$ $t > 0$ $\eta'(f_1, f_1, f_2,\ldots,f_n, t)$
$= \eta'(f_2, f_2, f_1, f_3,\ldots,f_n, t)$
$(I_9)\eta'(f_1,g,f_2,\ldots,f_n, t)$ is a monotonic
non-decreasing function of $\mathbb{C}$ and $\lim_{t\to\infty}\eta'(f_1, g, f_2, \ldots, f_n, t) = 1$
Then η is said to be the 2- fuzzy n-n inner product $F(X)^n$ and the pair $(F(X)^n, \eta')$ is called
2 - fuzzy n-n b-semi inner product space.

Definition 3.4: In a 2 fuzzy n-n b-inner product space, define $< h_i, h_j; d >_\alpha = \inf\{ t: \eta'(f_1, g, f_2, \ldots, f_n, t)\geq d\}$.

Definition 3.5: Let $(F(X^n),\eta)$ be a 2-fuzzy n-n inner product space . If $\{h_i\}$ are linearly independent in $F(X^n)$, then $\{h_i\}$ is said to be a b-orthogonal set if for $d \in F(X^n)$,
$< h_i, h_j; d >_\alpha=1$ if i=j
= 0 otherwise

Definition 3.6: Let $(F(X^n),<>_\alpha)$ be a 2-fuzzy n-n inner product space over K and d∈ $F(X^n)$ then

a) A sequence $\{f_n\}$ in $F(X^n)$ is said to be b-Cauchy sequence if for every $\varepsilon > 0$ there exists N>0 such that for every m,n ≥ N, $\|f_i - f_j, d, \ldots, f_n\| < \varepsilon$
b) $F(X^n)$ is said to be b-Hilbert if every b-Cauchy sequence is convergent in 2-fuzzy n-n-b semi inner product space.

Theorem 3.7(Bessel's Inequality): Let $(F(X^n),<>_\alpha)$ be a 2-fuzzy n-n-b inner product space over the scalar field K, then
$\sum_{i=1}^{n}|< h, g_i; d >_\alpha|^2 \leq \frac{1}{nu^{n+1}}\|h, d, f_3 \ldots, f_n\|_\alpha^2$ which holds for any h∈ $F(X^n)$ whenever
g_i,d∈ $F(X^n)$ such that $d \in span\{g_1,\ldots,g_n\}$and $< g_i, g_i; d >_\alpha = 0$ if i≠j and
$< g_i, g_i; d >_\alpha= 1$ if i=j

Proof: By definition,
$\|h, d, f_3 \ldots, f_n\|_\alpha^2= \inf\{t:\eta(h,d,d,f_3 \ldots, f_n,t^2) \geq \alpha\}$ and $d=\Sigma\,\beta_k\, g_k$
where $\beta_k=\inf\{t:\eta(h,d,g_k,f_3 \ldots, f_n,t^2)\geq\alpha\}$
Consider
$\|h, d, f_3 \ldots, f_n\|_\alpha^2 = \inf\{t:\eta(h,d,d,f_3 \ldots, f_n,t^2)\geq\alpha\}$
$= \inf\{ t:\eta(h, \Sigma\,\beta_k\, g_k , \Sigma\,\beta_k\, g_k,f_3 \ldots, f_n,t^2)\geq\alpha\} \geq nu^{n+1}\inf\{t:\inf\{t:\eta(h, \beta_k\, g_k, \beta_k\, g_k,f_3 \ldots, f_n,t^2)$
$\geq \alpha\}$
$= nu^{n+1} \inf\{t:\inf\{ t:\eta(h, g_k , g_k,f_3 \ldots, f_n,\frac{t^2}{|\beta_k|^2})$
$\geq \alpha\}$
$= nu^{n+1}\inf\{|\beta_k|^2 t:\inf\{ t:\eta(h, g_k$
$, g_k,f_3 \ldots, f_n,t^2)\geq\alpha\}$
Since $\eta(h, g_k , g_k,f_3 \ldots, f_n,t^2) = 1$
$\|h, d, f_3 \ldots, f_n\|_\alpha^2\geq nu^{n+1}|\beta_k|^2$

Since $\Sigma|\beta_k|^2$
$= \sup\{\inf\{t:\eta(h, g_k, g_k, f_3 \dots, f_n, t^2) \geq \alpha\}$
It follows that $\sum_{i=1}^{n}|< h, g_i; d >_\alpha|^2 \leq \frac{1}{nu^{n+1}}\|h, d, f_3 \dots, f_n\|_\alpha^2$

Theorem 3.8: If $\{g_i\}$ is a b-orthonormal set in a 2-fuzzy n-n-b-Hilbert space and if $h \in F(X^n)$ then the set $S = \{g_i :< h, g_i; d >_\alpha \neq 0\}$ is either empty or countable

Proof: Suppose $< h, g_i; d >_\alpha = 0$ then the set S is empty.If $< h, g_i; d >_\alpha \neq 0$ we need to prove that S is countable. For each positive integer n consider the set,
$S_n = \{g_i : |< h, g_i; d >_\alpha|^2 > \frac{\|h,d,f_3 \dots,f_n\|_\alpha^2}{n^2 u^{n+1}}\}$
By Bessel's inequality, $\sum_{i=1}^{n}|< h, g_i; d >_\alpha|^2 \leq \frac{1}{nu^{n+1}}\|h, d, f_3 \dots, f_n\|_\alpha^2$
If the set S_n contains n or more than n then, elements of $F(X^n)$ $\sum_{i=1}^{n}|< h, g_i; d >_\alpha|^2 > \frac{1}{nu^{n+1}}\|h, d, f_3 \dots, f_n\|_\alpha^2$ which contradicts Bessel's inequality.
Therefore S_n should have at most n-1 elements (i.e) S_n is a finite set.
By definition of S_n, $S = \cup_{n=1}^{\infty} S_n$.Since countable union of finite sets is countable S is countable.

Definition 3.9: Let $(F(X^n), \|., \dots, .\|_\alpha)$ be a 2-fuzzy b-n-n-normed space. Let W be a subspace of $F(X^n)$, and $d \in F(Xn)$ be fixed. Then a map $T:W\text{x}\langle d\rangle \to K$ is called a b-linear functional on $W\text{x}\langle d\rangle$ whenever for every $f,g \in W$ and $k \in K$,
(i) $T(f+g,d) = T(f,d) + T(g,d)$
(ii) $T(kf,d)=kT(f,d)$
A b-linear functional is said to be bounded if there exists a positive real number $M > 0$ such that
$|T(f, d)| \leq M\|f, d, f_3, \dots, f_n\|_\alpha$
It can be seen that
$\|T\| = \sup\{|T(f, d)|; \|f, d, f_3, \dots, f_n\|_\alpha \leq 1\}$
$\|T\| = \sup\{|T(f, d)|; \|f, d, f_3, \dots, f_n\|_\alpha = 1\}$
$\|T\| = \sup$
$\{|T(f, d)|/\|f, d, f_3, \dots, f_n\|_\alpha ; \|f, d, f_3, \dots, f_n\|_\alpha \neq 0\}$
And $|T(f, d)| \leq \|T\|\|f, d, f_3, \dots, f_n\|_\alpha$
For a fuzzy n-n-normed space $(F(X^n), \|., \dots, .\|_\alpha)$ and $d \in F(X^n)$,denote $F(X^n)_d^*$ to be the 2 fuzzy n-n Banach space of all bounded b-linear functionals on F(Xn) x⟨d⟩ where ⟨d⟩ is the subspace of $F(X^n)$ generated by d.

Theorem 3.10 (Riesz): Let $(F(X^n), <>_\alpha)$ be a 2-fuzzy n-n inner product space and $\{g_i\}$ be a b-orthonormal set in $F(X^n)$ and $k_i \in K$ then

1) If $\Sigma k_n g_n$ converges to some h in the 2-fuzzy n-n semi normed space $(F(X^n), \|., \dots, .\|_\alpha)$ then $< h, g_i; d >_\alpha = k_n$ for each n and $\Sigma|k_n|^2 < \infty$
2) If $F(X^n)$ is a 2-fuzzy n-n b-Hilbert space and $\Sigma|k_n|^2 < \infty$ then $\Sigma k_n g_n$ converges to some h in the 2 fuzzy n-n b-semi normed space $(F(X^n), \|., \dots, .\|_\alpha)$

Proof:

1) If $\Sigma k_n g_n$ converges to some h in $F(X^n)$, then $h = \Sigma k_n g_n$, since $\{g_i\}$ is a b-orthonormal set it is obvious that $< h, g_i; d >_\alpha = k_i$ for each i.

By Bessel's inequality $\sum_{i=1}^{n}|< h, g_i; d >_\alpha|^2 \leq \frac{1}{nu^{n+1}}\|h, d, f_3 \dots, f_n\|_\alpha^2$
Therefore $\Sigma|k_n|^2 < \infty$
2) For m=1,2,3,.. let $h_m = \sum_{n=1}^{m} k_n g_n$
For m>j ,$h_m - h_j = \sum_{n=j+1}^{m} k_n g_n$
We have $\|h_m - h_j, d, f_3 \dots, f_n\|_\alpha^2$
$= < h_m - h_j, h_m - h_j; d >_\alpha$
$= \sum_{n=j+1}^{m} k_n g_n < \infty$
Therefore $\{h_m\}$ is a b-Cauchy sequence in $F(X^n)$, and since $F(X^n)$ is a 2-fuzzy n-n Hilbert space, $\{h_m\}$ converges to some h in $F(X^n)$

Theorem 3.11: Let $\{g_\alpha\}$ be a b-orthonormal basis in a 2 fuzzy b-n-n Hilbert space $F(X^n)$, then for every h in $F(X^n)$, $h = \Sigma < h, g_n; d >_\alpha g_n$

Proof: Since $\{g_\alpha\}$ is a b-orthonormal basis in $F(X^n)$, $\{g_\alpha\}$ is a countable set. By Bessel's inequality we have, $\sum_{i=1}^{n}|< h, g_i; d >_\alpha|^2 \leq \frac{1}{nu^{n+1}} \|h, d, f_3 \dots, f_n\|_\alpha^2$
which implies that $\Sigma < h, g_n; d >_\alpha g_n$ converges to some f in $F(X^n)$,
(i.e) $f = \Sigma < h, g_n; d >_\alpha g_n$,
Also $< f, g_n; d >_\alpha = < \Sigma < h, g_n; d >_\alpha, g_n; d >_\alpha$
$= < h, g_n; d >_\alpha$
This implies $< h - f, g_n; d >_\alpha = 0$
Therefore (h-f) $\perp^d$ g_n for all n
If f≠h then let r = (h-f)/$\|h - f, d, f_3, \dots, f_n\|_\alpha$ which implies $\|r, d, f_3, \dots, f_n\|_\alpha$=1
Since(h-f) $\perp^d$ g_n for all n ,$< r, g_n; d >_\alpha = 0$
Therefore $\{g_\alpha\}\cup\{f\}$ is a b-orthonormal set in, which contradicts the maximality of the
b-orthonormal set $\{g_\alpha\}$. So f=h. Hence
$h = \Sigma < h, g_n; d >_\alpha g_n$

Theorem 3.12: Let (F(X^n) be a 2 fuzzy b-n-n Hilbert space and T∈ $F(X^n)^*_d$, then there exists a unique f ∈ $F(X^n)$ such that $T(h,d) = < h, f; d >_\alpha$ and $\|T\| = \|f, d, f_3, \dots, f_n\|_\alpha$

Proof: Let $\{g_i\}$ be a b-orthonormal set
For m=1,2,3,.. let$f_m = \sum_{n=1}^{m} T(g_n, d) g_n$
Since $\{g_i\}$ is a d-orthonormal set,
$\|f_m, d, f_3 \dots, f_n\|_\alpha^2 = \sum_{n=1}^{m}|T(g_n, d)|^2 = \beta_m$
Also $T(f_m,d) = \sum_{n=1}^{m}|T(g_n, d)|^2 = \beta_m$
Since T is bounded
$|T(g_n, d)| \leq \|f_m, d, f_3, \dots, f_n\|_\alpha$ which implies $\beta_m \leq \|T\|^2$
Letting m→ ∞, $\sum_{n=1}^{m}|T(g_n, d)|^2 \leq \|T\|^2 < \infty$
Let $\{g_\alpha\}$ be a b-orthonormal basis for $F(X^n)$.
Set $G_T = \{\{g_\alpha\}; T(g_\alpha, d) \neq 0$ and since G_T is countable let $G_T = \{g_1, g_2, \dots,\}$
Then $\Sigma|T(g_n, d)|^2 < \infty$. Therefore by Riesz theorem $\Sigma\overline{T(g_n, d) g_n}$ converges in $F(X^n)$.
Let $f = \Sigma\overline{T(g_n, d) g_n}$, we claim
$T(h,d) = < h, f; d >_\alpha$ for every h∈$F(X^n)$.
Let h∈ F(Xn), then $\{g_\alpha; < h, g_\alpha; d >_\alpha\}$ is countable.
Let it be $\{l_1, l_2, \dots,\}$. Then
$h = \Sigma < h, l_m; d >_\alpha l_m$, that implies
$T(h,d)=T(\Sigma h, l_m; d >_\alpha l_{m,} d)$
It is sufficient to show that
$T(h,d) = < l_m, f; d >_\alpha$ for m = 1,2,3,...
Fix m and
let$< l_m, f; d >_\alpha = \sum_n T(g_n,d) < l_m, g_n; d >_\alpha$.
If $g_{n_0} = l_m$ for some n_0,
Then $< l_m, f; d >_\alpha = T(g_{n_0}, d) = T(l_m,d)$
If $l_m \neq g_n$ for some n, then $< l_m, f; d >_\alpha = 0$
That implies $T(l_m,d)=0$
Therefore $T(l_m, d) = < h, f; d >_\alpha$ for all m
Hence $T(h,d) = < h, f; d >_\alpha$
To prove uniqueness
Let f_1, f_2 ∈$F(X^n)$ such that $T(h,d) = < h, f_1; d >_\alpha$ and $T(h,d) = < h, f_2; d >_\alpha$
That implies $< h, f_1 - f_2; d >_\alpha$=0 and
hence $f_2 = f_1$
In particular $< f_1 - f_2, f_1 - f_2; d >_\alpha = 0$ and hence $f_1 - f_2 = kd$ for some k∈K
Hence $f_1 - f_2 \in \langle d \rangle$
Therefore f is unique upto d-congruence
Now to show that $\|T\| = \|f, d, f_3, \dots, f_n\|_\alpha$
If T=0 then T(h,d) = 0 for all h and also $< h, f; d >_\alpha = 0$
Therefore f and d are linearly dependent and hence $\|f, d, f_3, \dots, f_n\|_\alpha = 0$ implies
$\|T\| = \|f, d, f_3, \dots, f_n\|_\alpha$

If T≠ 0 then T(h,d) ≠ 0 and $< h, f; d >_\alpha \neq 0$
Therefore f≠ 0 or f and d are linearly dependent
Hence $\|f, d, f_3 \ldots, f_n\|_\alpha^2 = < f, f; d >_\alpha$
$= T(f,d) \leq \|T\| \|f, d, f_3, \ldots, f_n\|_\alpha$
Therefore
$\|f, d, f_3, \ldots, f_n\|_\alpha \leq \|T\|$ (1)
We also have
$T(h,d) = |< h, f; d >_\alpha|$
$\leq \|h, d, f_3, \ldots, f_n\|_\alpha \|f, d, f_3, \ldots, f_n\|_\alpha$
Hence $\|T\| = \sup\{|T(h, d)|; \|h, d, f_3, \ldots, f_n\|_\alpha = 1\}$
$= \sup \{|< h, f; d >_\alpha|\}$
$\leq \|f, d, f_3, \ldots, f_n\|_\alpha$ (2)
From (1) and (2) $\|T\| = \|f, d, f_3, \ldots, f_n\|_\alpha$

References:

1. T.Bag, Samanta S.K, 2003, “Finite Dimensional fuzzy normed linear spaces”, J.Fuzzy.Math.11. no.3 687-705
2. S.C.Cheng, Mordenson J.N, 1994, “Fuzzy linear operators and fuzzy normed l inear spaces”, Bull.Cal.Math.Soc.86,(2003) 429-436
3. Y.J.Cho, M Matic and J.Pecaric, Inequalities of Hlawka’s type in n-inner product spaces, Commun, Korean Math.Soc. 17(2002), no. 583-592
4. C.Dimminie, S.Gahler and A.White, 2-inner product spaces, Demonstratio Math 6(1973), 525-536
5. A.M.El-Abyad and H.M. El-Hamouly, Fuzzy inner product spaces, Fuzzy Sets and Systems 44(1991) no.2, 309-326.
6. C.Felbin, Finite Dimensional fuzzy normed linear space, Fuzzy sets and systems 48(1992) no.2, 239-248
7. _______, The completion of a fuzzy normed linear space, J.Math.Anal.Appl.174(1993), no.2 428-440
8. ______, Finite dimensional fuzzy normed linear space II, J.Anal.7(1999),117-131
9. S.Gahler 1964, “Linear 2-normierte Raume”, Math Nachr,28 1-43
10. P. K. Harikrishnan, P. Riyas and K. T. Ravindran, Riesz Theorems in 2-InnerProduct spaces
11. HendraGunawan, Mashadi, Sri Gemawati, Nursupiamin, Idha Sihwaningrum, Orthogonality in 2-normed spaces revisited, Univ.Beograd.Publ.Elektrotehin.FakSer Mat.17(2006), 76-83
12. A.K.Katsaras 1984, “fuzzy topological vector space”, fuzzy set and system 12,143-154
13. J.K.Kohli and R.Kumar, On fuzzy inner product spaces and fuzzy co-inner product spaces, Fuzzy sets and systems 53(1993) no.2, 227-232
14. ________ , Linear mappings ,fuzzy linear spaces, fuzzy inner product and fuzzy co-inner product spaces, Bull.Calcutta Math.Soc.87(1995), no.3 237-246
15. S.V.Krishna and K.K.M.Sharma, Separation of fuzzy normed linear spaces, Fuzzy sets and systems 63(1994) no.2, 207-217
16. G.S.Rhie, B.M.Choi and D.S.Kim, On the completeness of fuzzy normed linear spaces, Math.Japan 45(1997), no.1, 33-37
17. R.M Somasundaram and Thangaraj Beaula,Some aspect of Fuzzy 2- Noemed linear spaces. Bull.Malyasian Math. Sci.Soc,.(2009) 32:211-222,.
18. Srinivasan Vijayabalaji and Natesan Thillaigovindan, Fuzzy n-inner product space, Bull,KoreanMath.Soc. 43(2007), No.3.pp,447-459
19. Thangaraj Beaula and ChristinalGunaseeli, Fixed point theorem for 2- fuzzy n - b metric spaces Annals of Fuzzy Mathematics and Informatics Volume x, No. x, (Month 2014), pp. 1–xx ISSN: 2093–9310 (print version) ISSN: 2287–6235 (electronic version)
20. Thangaraj Beaula and Daniel Evans, Some Aspects of 2-Fuzzy n-n Normed Linear Space International Journal of Applications of Fuzzy Sets and Artificial Intelligence (ISSN 2241-1240), Vol. 7 (2017), 23-44
21. 21.Thangaraj Beaula and R.A.S.Gifta, On complete 2-fuzzy dual normed linear spaces, Journal of Advanced Studies in Topology, 4(2) (2013) 34-42
22. L.A Zadeh,1965 “Fuzzy Sets”, Information and Control 8, 338-353
23. Mariapresenti. L, Arockiarani. I, Vague Relation With Max- Min Composition Technique In Medical Diagnosis; Mathematical Sciences International Research Journal : Issn 2278-8697 Volume 6 Issue 1 (2017), Pg 110-113

***