EDGE - VERTEX MIXED DOMINATION ON S–VALUED GRAPHS

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Abstract: Chandramouleeswaran et.al. [8] studied a semiring valued graphs and have proved several results on that. The notion of domination on vertex sets and edge sets in the $S-$ valued graph G^S have been discussed in [3] and [5]. In our earlier paper [6], we studied the vertex - edge mixed domination on $S-$ valued graphs. In this paper we study the concept of edge - vertex mixed domination on $S-$ valued graphs.

Keywords: Semirings, Graphs, $S-$Valued Graphs, $Ev-$ Weight $M-$ Dominating Set.

AMS Classification: 05C25, 16Y60.

Introduction: The notion of semiring was first introduced by H.S.Vandiver[9], in the year 1934. Jonathan Golan [2], in his book mentioned the notion of $S-$ valued graph. But nothing more has been dealt on this concept. Motivated by this, Chandramouleeswaran et.al.[8] studied the $S-$ Valued graph in detail.
The theory of domination of graphs was first developed by Berge [1]. Several author's discussed the theory of domination and have arrived several bounds for the domination number of a graph. In our earlier paper [6], we studied the vertex - edge mixed domination on $S-$valued graphs. In this paper we study the concept of edge - vertex mixed domination on $S-$valued graphs.

Preliminaries: In this section, we recall some basic definitions that are needed for our work.

Definition: 2.1. [2] *A semiring$(S,+,\cdot)$ is an algebraic system with a non-empty set S together with two binary operations + and $\cdot$ such that*

1. (S,+,0) is a monoid.
2. (S,$\cdot$) is a semigroup.
3. For all a,b,c ∈ S , a $\cdot$ (b + c) = a $\cdot$ b + a $\cdot$ c and (a + b) $\cdot$ c = a $\cdot$ c + b $\cdot$ c.
4. 0 $\cdot$ x = x $\cdot$ 0 = 0 ∀ x ∈ S.

Definition: 2.2. [2] *Let $(S,+,\cdot)$ be a semiring. A Canonical Pre-order in S defined as follows: for $a,b \in S$, $a \preccurlyeq b$ if and only if, there exists an element $c \in S$ such that $a + c = b$.*

Definition: 2.3. [1] *A subset $F \subseteq E$ of edges in a graph $G = (V,E)$ is called an edge dominating set in G if for every edge $e \in E - F$ there exist an edge $f \in F$ such that e and f have a vertex in common.*

Definition: 2.4. [1] *A subset $M \subseteq E$ is an Independent edge set of G if f, g∈ M,*
$N(f) \cap \{g\} = \varphi$.

Definition: 2.5. [1] *A subset $M \subseteq E$ is an Independent edge dominating set of G if M is both an independent edge set and a dominating edge set.*

Definition: 2.6.[8] *Let $G = (V, E \subset V \times V)$ be a given graph with $V, E \neq \phi$. For any semiring $(S,+,\cdot)$, a semiring-valued graph (or a S–valued graph), G^S, is defined to be the graph $G^S = (V,E,\sigma,\psi)$ where $\sigma: V \to S$ and $\psi : E \to S$ are functions defined as follows:*

$$\psi(x,y) = \begin{cases} \min\{\sigma(x),\sigma(y)\} & if\ \sigma(x) \preceq \sigma(y)\ or\ \sigma(y) \preceq \sigma(x) \\ 0 & otherwise \end{cases}$$

for every unordered pair (x,y) of $E \subset V \times V$. *We call* σ, *a S–vertex set and* ψ, *a S–edge set of* G^S.

Definition: 2.7. [4] *A S–valued graph* $G^S = (V,E,\sigma,\psi)$ *is said to be a S–Star(S–Wheel) if its underlying graph G is a Star(Wheel) along with S–values.*

Definition: 2.8. [7] *Let* $G^S = (V,E,\sigma,\psi)$ *be a S–valued graph. Let* $e \in E$ *.The open neighbourhood of e, denoted by* $N_S(e)$, *is defined to be the set* $N_S(e) = \{(e_i,\psi(e_i)) \,/\, e \text{ and } e_i \in E \text{ are adjacent}\}$
The closed neighbourhood of e, denoted by $N_S[e]$, *is defined to be the set* $N_S[e] = N_S(e) \cup (e,\psi(e))$

Definition: 2.9. [5] *Let* $G^S = (V,E,\sigma,\psi)$ *be a S–valued graph. A subset* $M \subseteq E$ *is an independent edge set of* G^S *if f, g*$\in$ *M such that* $N_S(f) \cap (g,\psi(g)) = \varphi$.

Definition: 2.10. [5] *Let* $G^S = (V,E,\sigma,\psi)$ *be a S–valued graph. A subset* $M \subseteq E$ *is said to be a minimal independent edge set if*

1. M is an independent edge set.
2. No proper subset of M is an independent edge set.
3.

Definition: 2.11. [5] *Let* $G^S = (V,E,\sigma,\psi)$ *be a S–valued graph. A subset* $M \subseteq E$ *is said to be a maximal independent edge set if*

1. M is an independent edge set.
2. If there is no subset M' of E such that $M \subset M' \subset E$ and M' is an independent edge set.

Definition: 2.12. [6]*Consider the S–valued graph* $G^S = (V,E,\sigma,\psi)$. *A vertex* $v \in V$ *is said to be a ve–weight m–dominating vertex of an edge e , if* $\psi(e) \preceq \sigma(v) \; \forall \in \; v \in \langle N_S[e] \rangle$

Definition: 2.13. [6]*Consider the S–valued graph* $G^S = (V,E,\sigma,\psi)$. *Let* $D \subseteq V$. *If every edge of* G^S *is weight m–dominated by any vertex in D, then D is said to be a ve–weight m–dominating set.*

Edge - Vertex Mixed Domination on S–Valued Graphs: In this section, we introduce the notion of edge - vertex mixed domination in *S*–valued graph, analogous to the notion in crisp graph theory, and prove some simple results.

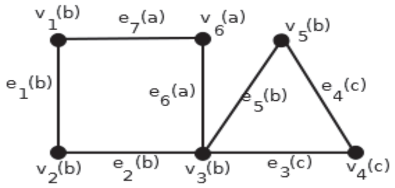
Definition: 3.1. *Consider the S–valued graph* $G^S = (V,E,\sigma,\psi)$. *Let* $e \in E$, *by definition* $N_S[e] = \{(f,\psi(f)), \text{where } e \text{ and } f \text{ are adjacent}\} \cup \{(e,\psi(e))\}$. *The induced subgraph of* $N_S[e]$, *denoted by* $\langle N_S[e] \rangle$ *is defined as* $\langle N_S[e] \rangle = (P \times S, N_S[e])$, *where* $P = \{v/v \text{ is an end point of } e \in N_S[e]\}$

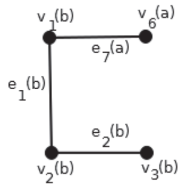
Example: 3.2. Let $(S = \{0,a,b,c\},+,\cdot)$ be a semiring with the following Cayley Tables:

+	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	b
c	c	c	b	c

.	0	a	b	c
0	0	0	0	0
a	0	0	a	0
b	0	a	b	c
c	0	0	c	c

Let be a canonical pre-order in *S*, given by
$0 \preceq 0, 0 \preceq a, 0 \preceq b, 0 \preceq c, a \preceq a, a \preceq b, a \preceq c, b \preceq b, c \preceq b, c \preceq c$
Consider the *S*– valued graph $G^S = (V,E,\sigma,\psi)$ and $\langle N_S[e_1] \rangle = (\{(v_1,b),(v_2,b),(v_3,b),(v_6,a)\},\{(e_1,b),(e_2,b),(e_7,a)\})$





G^S

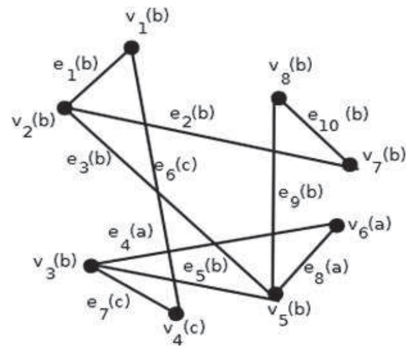
Definition: 3.3. *Consider the S–valued graph $G^S = (V,E,\sigma,\psi)$. An edge $e \in E$ is said to be a ev–weight m–dominating edge of a vertex v , if* $\sigma(v) \preceq \psi(e),\ \forall\, v \in \langle N_S[e] \rangle$.

In example 3.2, the edge e_1 is a *ev*– weight *m*– dominating edge of the vertices v_1, v_2, v_3 and v_6.

Definition: 3.4. *Consider the S–valued graph $G^S = (V,E,\sigma,\psi)$. Let $T \subseteq E$. If every vertex of G^S is weight m–dominated by any edge in T, then T is said to be a ev–weight m–dominating set.*
In example 3.2, $T = \{e_1, e_5\}$ is a *ev*– weight *m*– dominating set.

Definition: 3.5. *Consider the S–valued graph $G^S = (V,E,\sigma,\psi)$. A subset $T \subseteq E$ is said to be a minimal ev–weight m–dominating set, if*
1. T is a ev–weight m–dominating set.
2. No proper subset of T is a ev–weight m–dominating set.

Example: 3.6. Let $(S = \{0,a,b,c\},+,\cdot)$ be a semiring with the canonical preorder given in example 3.2
Consider the *S*– valued graph $G^S = (V,E,\sigma,\psi)$:



Clearly $T_1 = \{e_1,e_2,e_3,e_5,e_9,e_{10}\}$, $T_2 = \{e_1,e_3\}$, $T_3 = \{e_1,e_5\}$, $T_4 = \{e_1,e_9\}$,
$T_5 = \{e_2,e_5\}$, $T_6 = \{e_3,e_5\}$, $T_7 = \{e_1,e_2,e_3\}$, $T_8 = \{e_1,e_3,e_5\}$, $T_9 = \{e_1,e_3,e_9\}$, $T_{10} = \{e_1,e_3,e_{10}\}$.... are all *ev*– weight *m*– dominating sets.
$\therefore$ $T_2 = \{e_1,e_3\}$, $T_3 = \{e_1,e_5\}$, $T_4 = \{e_1,e_9\}$, $T_5 = \{e_2,e_5\}$, $T_6 = \{e_3,e_5\}$, are all minimal *ev*– weight *m*– dominating sets.
From the above example, we can observe that a minimal *ev*– weight *m*– dominating set in a *S*–valued graph G^S need not be unique; however the cardinality of all the sets is the same.

Definition: 3.7. *Consider the S–valued graph $G^S = (V,E,\sigma,\psi)$. A subset $T \subseteq E$ is said to be a maximal ev–weight m–dominating set, if*
1. T is a ev–weight m–dominating set.
2. there is no ev–weight m–dominating set $T' \subset E$ such that $T \subset T' \subset E$.

In example 3.6, $T_1 = \{e_1,e_2,e_3,e_5,e_9,e_{10}\}$ is a maximal *ev*– weight *m*– dominating set.
while observing example3.2, we conclude that a maximal *ev*– weight *m*–dominating set need not be unique.

Definition: 3.8. *Consider the S–valued graph $G^S = (V,E,\sigma,\psi)$. A subset $T \subseteq E$ is said to be a ev–weight m–dominating independent set, if*
1. T is a ev–weight m–dominating set.
2. If $e, f \in T$ then $NS(e) \cap (f,\psi(f)) = \phi$..

In example 3.2, $T = \{e_1,e_5\}$ is a *ev*– weight *m*– dominating set. Also $N_S(e_1) \cap \{(e_5,b)\} = \varphi$. Hence $T = \{e_1,e_5\}$ is a *ev*– weight *m*– dominating independent set.

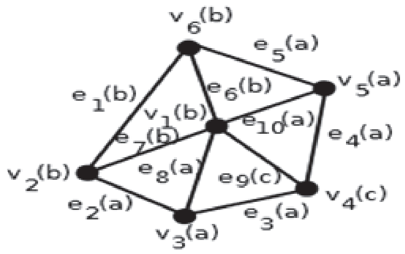
Definition: 3.9. *Consider the S–valued graph $G^S = (V,E,\sigma,\psi)$. Let $T \subseteq E$ be a weight dominating edge set of G^S. An edge $e \in T$ is said to be an S –isolate edge if $N_S(e) \subseteq (E - T) \times S$.*
In example 3.2, $T = \{e_1,e_5\}$, here the edges e_1 and e_5 are the isolate edges, since $N_S(e_1) = \{(e_2,b),(e_7,a)\} \subseteq (E - T) \times S$, and $N_S(e_5) = \{(e_2,b),(e_3,c),(e_4,c),(e_6,a)\} \subseteq (E - T) \times S$.

Theorem: 3.10. *The minimal ev–weight m–dominating set of a S–Star will be an edge with maximum weight.*
Proof: *Let G^S be a S–Star.*
Let $e_1 \in G^S$ be an edge with maximum weight.
Then $v_i \in \langle N_S[e_1] \rangle$, $\forall v_i \in G^S$.
Also $\psi(e_1) \succeq \sigma(v_i), \quad \forall v_i \in G^S$.
∴ The edge e_1 will weight m–dominate all the vertices of G^S.
Hence the minimal ev–weight m–dominating set of G^S is $\{e_1\}$.
Analogously we can prove the following theorems.

Theorem: 3.11.
1. The minimal ev–weight m–dominating set of a complete graph will be an edge with maximum weight.
2. The minimal ve–weight m–dominating set of a S–Wheel will be an edge with maximum weight, if the edge is a spoke of the S–Wheel.

Remark: 3.12. The minimal *ve*– weight *m*– dominating set of a*S*–Wheel will not be an edge with maximum weight, if the edge is not a spoke of the *S*–Wheel.

Example: 3.13. Let $(S = \{0,a,b,c\},+,\cdot)$ be a semiring with the canonical preorder given in example 3.2
Consider the *S*– valued graph $G^S = (V,E,\sigma,\psi)$:



Here the edge e_1 is a *ev*– weight *m*– dominating edge of the vertices $v_1,v_2,v_3,v_5$and v_6. The vertex v_4 is not weight *m*– dominated by the edge e_1. Hence $\{e_1\}$ is not a minimal *ve*– weight *m*– dominating set, since e_1 is not a spoke of the *S*–Wheel.

Theorem: 3.14. *A ev–weight m–dominating set T of a graph G^S is a minimal ev–weight m–dominating set of G^S iff every edge $e \in T$ satisfies at least one of the following properties:*
1. there exist an edge f ∈ E –T, such that NS(f) ∩ (T × S) = {(e,ψ(e))}
2. e is adjacent to no edge of T.

Proof: Let $e \in T$. Assume that e is adjacent to no edge of T, then $T - \{e\}$ cannot be a *ev*– weight *m*– dominating set. ⇒ T is a minimal *ev*– weight *m*– dominating set.
On the other hand, if for any $e \in T$ there exist a $f \in E - T$ such that
$N_S(f) \cap (T \times S) = \{(e,\psi(e))\}$
Then f is adjacent to $e \in T$ and no other edge of T.
In this case also, $T - \{e\}$ cannot be a *ev*– weight *m*– dominating set of G^S.
Hence T is a minimal weight dominating edge set.
Conversely, assume that T is a minimal *ev*– weight *m*– dominating set of G^S. Then for each $e \in T$, $T - \{e\}$ is not a *ev*– weight *m*– dominating set of G^S.
∴ there exist an edge, $f \in E - (T - \{e\})$ that is adjacent to no edge of $(T - \{e\})$.
If $f = e$, then e is adjacent to no edge of T .
If $f \neq e$, then T is a *ev*– weight *m*– dominating set and f $T \Rightarrow f$ is adjacent to atleast one edge of T. However f is not adjacent to any edge of $T - \{e\}$.
$\Rightarrow N_S(f) \cap T \times S = \{(e,\psi(e))\}$.

Theorem: 3.15. *A subset $T \subseteq E$ of G^S is a ev–weight m–do∉minating independent set iff T is a maximal independent edge set in G^S.*

Proof: Clearly every maximal independent edge set T in G^S is a *ev*– weight *m*– dominating independent set.
Conversely, assume that T is a *ev*– weight *m*–dominating independent set. Then T is independent and every edge not in T is adjacent to an edge of T and therefore T is a maximal independent edge set in G^S.
Theorem: 3.16. *Every maximal independent edge set of G^S is a minimal ev–weight m–dominating set.*
Proof: *Let T be a maximal independent edge set of G^S. Then by theorem 3.15, T is a ev–weight m–dominating set. Since T is independent, every edge of T is adjacent to no edge of T. Thus, every edge of T satisfies the second condition of theorem 3.14. Hence T is a minimal ev–weight m–dominating set in G^S.*
Combining the above two theorems, we obtain the following theorem,

Theorem: 3.17. *A subset $T \subseteq E$ of G^S is a ev–weight m–dominating independent set iff T is a minimal ev–weight m–dominating set.*

Theorem: 3.18. *Let G^S be a vertex regular S–valued graph. If $T \subseteq E$ of G^S is a minimal ev–weight m–dominating set without S–isolate edges then $E - T$ is also a ev–weight m–dominating set of G^S.*
Proof: *Assume that $G^S = (V,E,\sigma,\psi)$ be a vertex regular S–valued graph.*
Let $T \subseteq E$ be a minimal ev–weight m–dominating set.
Let e ∈ T, then by theorem 3.14 ,
1. there exist an edge f ∈ E –T, such that
$NS(f) \cap (T \times S) = \{(e,\psi(e))\}$
2. e is adjacent to no edge of T.
In the first case, e is adjacent to some edge in E –T .
In the second case, e is an S–isolate edge of the subgraph spanned by .
But e is not S–isolated in GS.
Hence e is adjacent to some edge of E –T.
Thus E –T is aev–weight m–dominating set of GS, whenever GS is vertex regular S–valued graph.

Remark: 3.19. In the above theorem, the vertex regularity of G^S is essential. That is, if the graph G^S is not vertex regular then the theorem fails as given by the following example.
In example 3.6, $T_2 = \{e_1, e_3\}$, is a minimal *ev*– weight *m*– dominating set without *S*– isolate edges. And $E - T_2 = \{e_2, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$.
Since the edges of $E - T_2$ have minimum weight, $E - T_2$ is not a *ev*– weight *m*– dominating set.

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