FRACTIONAL KINETIC EQUATIONS INVOLVING STRUVE FUNCTION USING SUMUDU TRANSFORM

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Abstract: The importance of fractional differential equations in the field of applied science gained more attention not only in mathematics but also in physics and engineering applications. The effectiveness and the importance of the kinetic equation in certain astrophysical problems we develop a generalized form of the fractional kinetic equation involving Struve functions. The obtained results are useful to investigate many problems in Mathematical physics.

Keywords: Fractional Calculus, Kinetic Equations, Mittag-Leffler Function, Sumudu Transform.

Introduction: The Struve function [1]

$$H_v(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{z}{2}\right)^{2n+v+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+v+\frac{3}{2}\right)}, \tag{1}$$

is a particular solution of the non-homogeneous Bessel differential equation

$$xy''(x)+xy'(x)+\left(x^2-v^2\right)y(x)=\frac{4\left(\frac{x}{2}\right)^{2k+v+1}}{\sqrt{\pi}\,\Gamma\left(v+\frac{1}{2}\right)} \tag{2}$$

where Γ is the classical gamma function. The function $z^{-v-1}H_v(z)$ is entire functions of z and v. Recently, Watugala [2,3] introduced Sumudu integral transform is defined as follows (see [4–6]):

$$G(u)=S\left[f(t);u\right]=\int_0^{\infty}e^{-t}f(ut)\,dt, \tag{3}$$

for $u\in(-\tau_1,\tau_2)$ where,

$$\left|f(t)\right|<\begin{cases}Me^{-t/\tau_1}, t\le 0\\ Me^{t/\tau_2}, t\ge 0\end{cases},$$

and M,τ_1,τ_2 are some positive real constants.

The generalized Mittag-Leffler function $E_{\alpha,\beta}(x)$ is defined by (see [7]):

$$E_{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\alpha n+\beta)} \tag{4}$$

Recent studies observed that the solutions of fractional order differential equations could model real-life situations better, particularly in reaction-diffusion type problems. Due to the potential applicability to wide variety of problems, fractional calculus is developed to large area of Mathematics physics and other engineering applications [8]-[15]. In view of the effectiveness and a great importance of the kinetic equation in certain astrophysical problems, the author develop a further generalized form of the fractional kinetic equation involving Struve function using Sumudu transform method.

Solutions of Generalized Fractional Kinetic Equation for Struve Function: In this section, we will investigate the solution of the generalized fractional kinetic equations by considering Struve function. The results are as follows.

Theorem 1: If $d > 0, v > 0, \mu, t \in C$ and $\mu > -\frac{3}{2}$ then the solution of fractional kinetic equation

$$N(t) = N_0 H_\mu(d^v t^v) - d^v D_t^{-v} N(t) \tag{5}$$

is given by

$$N(t) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma[v(2n+\mu+1)+1]}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\mu+\frac{3}{2}\right)} \times \frac{1}{t}\left(\frac{d^v t^v}{2}\right)^{2n+\mu+1} E_{v,v(2n+\mu+1)}(-d^v t^v) \tag{6}$$

Proof: The Sumudu transform of Riemann-Liouville fractional integral operator is given by

$$S\{{}_0D_t^{-v} f(t); u\} = u^v G(u) \tag{7}$$

where $G(u)$ is defined in (3).

Now, applying the Sumudu transform to both sides of (5) and applying (1) and using

$$S\{{}_0D_t^{-v} f(t)\} = S\left\{\frac{t^{v-1}}{\Gamma(v)}\right\} S\{f(t)\} = u^v G(u)$$

We have,

$$\begin{aligned} N^*(u) &= S[N(t);u] = N_0 S[H_\mu(d^v t^v);u] \\ &\quad - d^v S[{}_0D_t^{-v} N(t);u] \\ &= N_0\left[\int_0^\infty e^{-pt} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{d^v u^v t^v}{2}\right)^{2n+\mu+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\mu+\frac{3}{2}\right)} dt\right] \\ &\quad - d^v u^v N^*(u) \end{aligned} \tag{8}$$

where

$$S\{t^{\lambda-1}\} = u^{\lambda-1}\Gamma(\mu)$$

Then we get,

$$N^*(u)[1+d^v u^v] = N_0 \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{d^v}{2}\right)^{2n+\mu+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\mu+\frac{3}{2}\right)} \times \int_0^\infty e^{-pt} (ut)^{v(2n+\mu+1)} dt$$

$$= N_0 \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma[v(2n+\mu+1)+1]}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\mu+\frac{3}{2}\right)} \left(\frac{u^v d^v}{2}\right)^{2n+\mu+1}$$

Therefore

$$N^*(u) = N_0 \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma[v(2n+\mu+1)+1]}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\mu+\frac{3}{2}\right)} \times \left\{u^{v(2n+\mu+1)} \sum_{r=0}^{\infty} -(du)^r\right\} \tag{9}$$

Taking inverse s Sumudu transform of (9) and using $S^{-1}\left\{u^{v};t\right\}=\dfrac{t^{v-1}}{\Gamma(v)},\Re(v)>0$

we have

$$S^{-1}\left\{N^{*}(u)\right\}=N_0\sum_{n=0}^{\infty}\frac{(-1)^n\,\Gamma\left[v(2n+\mu+1)+1\right]}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\mu+\frac{3}{2}\right)}$$

$$\times S^{-1}\left\{\sum_{r=0}^{\infty}(-1)^r(d)^{vr}u^{v(2n+\mu+r+1)}\right\}$$

Which gives

$$N(t)=N_0\sum_{n=0}^{\infty}\frac{(-1)^n\,\Gamma\left[v(2n+\mu+1)+1\right]}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\mu+\frac{3}{2}\right)}\frac{1}{t}\left(\frac{d^v t^v}{2}\right)^{2n+\mu+1}$$

$$\times\left\{\sum_{r=0}^{\infty}(-1)^r(d)^{vr}\frac{t^v}{\Gamma\left(v(2n+\mu+r+1)\right)}\right\}$$

In view of definition of Mittag-Leffler function given in (4), we obtained the required result.

Theorem 2:

If $\boldsymbol{a}>0, d>0, v>0, \mu, t\in$ □ $, \boldsymbol{a}\neq d$ and $\mu>-\frac{3}{2}k$ then the solution of fractional kinetic equation

$$N(t)=N_0H_{\mu}\left(d^v t^v\right)-\boldsymbol{a}^v D_t^{-v}N(t) \tag{10}$$

is given by

$$N(t)=\sum_{n=0}^{\infty}\frac{(-1)^n\,\Gamma\left[v(2n+\mu+1)+1\right]}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\mu+\frac{3}{2}\right)}\times\frac{1}{t}\left(\frac{d^v t^v}{2}\right)^{2n+\mu+1}E_{v,v(2n+\mu+1)}\left(-\boldsymbol{a}^v t^v\right) \tag{11}$$

Proof: The proof of Theorems 2 would run parallel to those of Theorem 1.

Graphical Representations: In this section we plot the graphs of main results established in (6). Graphs of the solution of Eq. (6) are depicted below for some parameter values, that is $N_0=\mu=d=1$ and different values of v.

For Fig. 1, 2 and 3, we choose $v=0.1,0.2,0.3,0.4;0.5,0.7,0.9,1,1.5;$ and $1.6,1.7,1.8,1.9$ respectively

Fig 1: Solution of fractional kinetic equation (6) for
$v=0.1,0.2,0.3,0.4$

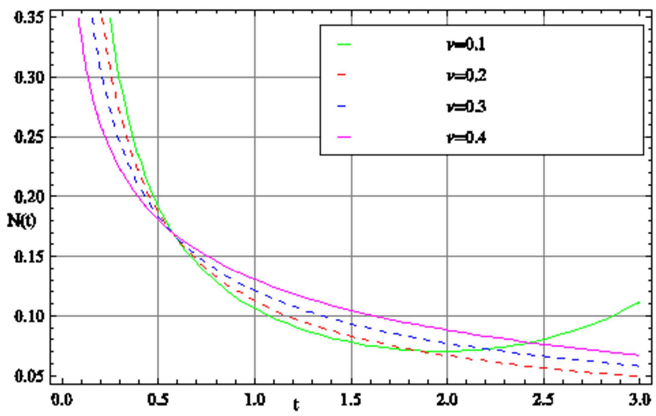


Fig 2: Solution of fractional kinetic equation (6) for $v = 0.5, 0.7, 0.9, 1, 1.5$

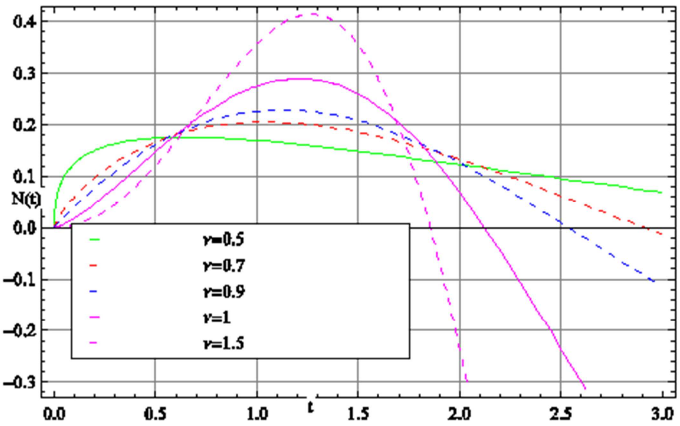
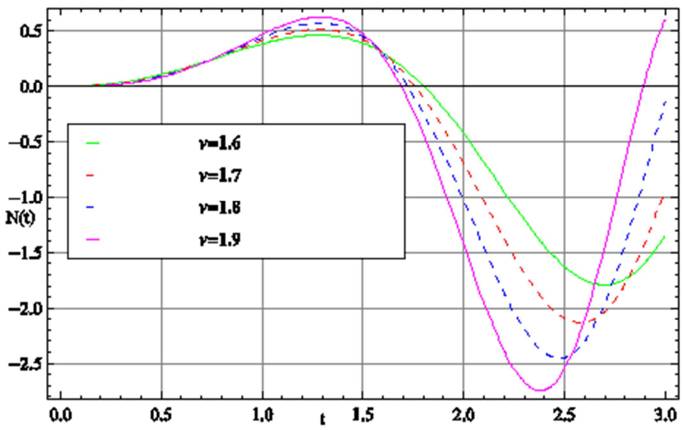


Fig 3: Solution of fractional kinetic equation (6) for $v = 1.6, 1.7, 1.8, 1.9$



Conclusion: In this paper, we have established solution of fractional kinetic equation involving Struve function with the help of Sumudu transform. It is not difficult to obtain several further analogous fractional kinetic equations and their solutions as those exhibited here by Theorem 1 and 2.

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