LATTICE CONSTRUCTION ON PRE A* -ALGEBRA

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Abstract: This paper analyzes the notion of lattice structure on Pre A*-algebra. It has been derived the corresponding properties of the Pre A*-lattice L. Furthermore, identified a congruence relation θ_a on L and proved that the set of all congruences on L is a distributive Pre A*-Lattice. Also described an ideal on Pre A*-lattice L and shown that F(L) the set of all ideals of L is a distributive Pre A*-lattice under the set inclusion. Also introduced the notion of ideal congruence on Pre A*-lattice and derived its various significant properties.

Key Words: A*-Algebra, Pre-A*-Algebra, Boolean Algebra, Partially Ordered Set, Ada, Homomorphism.

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Introduction: The concept of lattice was thoroughly discussed by Birkoff G (1948). In a draft manuscript entitled "The Equational theory of Disjoint Alternatives", E. G. Manes (1989) introduced the concept of Ada (Algebra of disjoint alternatives) $(A, \wedge, \vee, (-)', (-)_\pi, 0, 1, 2)$ which is however differs from the definition of the Ada of E. G. Manes (1993) later paper entitled "Adas and the equational theory of if-then-else". While the Ada of the earlier draft seems to be based on extending the If-Then-Else concept more on the basis of Boolean algebras and the later concept is based on C-algebras $(A, \wedge, \vee, (-)^\sim)$ introduced by Fernando Guzman and Craig C. Squir (1990) . P. Koteswara Rao (1994) first introduced the concept of A*-algebra $(A, \wedge, \vee, *, (-)^\sim, (-)_\pi, 0, 1, 2)$ not only studied the equivalence with Ada, C-algebra, Ada's connection with 3-Ring, Stone type representation but also introduced the concept of A*-clone, the If-Then-Else structure over A*-algebra and Ideal of A*-algebra. Later Ramabhadra Sarma.I., Koteswara Rao.P (1996) made an account on the algebra of Disjoint Alternatives.

J.Venkateswara Rao (2000) introduced the concept Pre A*-algebra $(A, \wedge, \vee, (-)^\sim)$ analogous to C-algebra as a reduct of A*- algebra. Srinivasa Rao.K.(2009) contributed the structural compatibility of Pre A*-Algebra with Boolean algebra. Further A. Satyanarayana (2012) established the concept of Ideals, Semilattice structures and Ideal congruences on Pre A*-algebra. Boolean algebra depends on two element logic. C-algebra, Ada, A*-algebra and our Pre A*-algebra are regular extensions of Boolean logic to 3 truth values, where the third truth value stands for an undefined truth value. The Pre A*- algebra structure is denoted by $(A, \wedge, \vee, (-)^\sim)$ where A is non-empty set $\wedge$, $\vee$, are binary operations and $(-)^\sim$ is a unary operation.

In this paper we identify for any subset L of a Pre A*-algebra, a Pre A*-lattice. We present various examples of Pre A*-lattices. We offer several properties of Pre A*-lattices. We define sub Pre A*-lattice, distributive Pre A*-lattices and homomorphism of Pre A*-lattices. We confer congruence relation θ_a on L and prove that the set of all congruences of the form θ_a forms a distributive Pre A*-Lattice. We also introduce the concept of Ideal, Ideal congruences on Pre A*-lattice and derived some important properties of these.

Preliminaries: In this section we concentrate on the algebraic structure of Pre A*-algebra and state some results which will be used in the later text.

Definition: The relation R on a set A is called a partial order on A when R(≤) is reflexive, anti-symmetric, and transitive. Under these conditions, the set A is called a partially ordered set or a poset. Frequently we write (A, R) or (A, ≤) to denote that A is partially ordered by the relation R(≤). Since the relation ≤ on the set of real numbers is the prototype of a partial order it is common to write ≤ to represent an arbitrary partial order can be described as follows:

1. For all $a \in A$, $a \le a$ (symmetry)
2. For all $a, b \in A$, $a \le b$, $b \le a$, then $a = b$ (anti symmetry)
3. For all $a, b, c \in A$, $a \le b$ and $b \le c$, then $a \le c$ (transitivity)

Two elements a and b in A are said to be comparable under ≤ if either $a \le b$ or $b \le a$; otherwise they are incomparable. If every pair of elements of A are comparable, then we say that the partially ordered set is totally ordered.

Definition: An algebra $(A, \wedge, \vee, (-)^{\sim})$ where A is a non-empty set with 1, $\wedge$, $\vee$ are binary operations and $(-)^{\sim}$ is a unary operation satisfying

(a) $x^{\sim\sim} = x \quad \forall x \in A$

(b) $x \wedge x = x, \quad \forall x \in A$

(c) $x \wedge y = y \wedge x, \quad \forall x, y \in A$

(d) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim} \quad \forall x, y \in A$

(e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \quad \forall x, y, z \in A$

(f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad \forall x, y, z \in A$

(g) $x \wedge y = x \wedge (x^{\sim} \vee y), \quad \forall x, y \in A$ is called a Pre A*-algebra.

Example: **3** = {0, 1, 2} with operations $\wedge$, $\vee$, $(-)^{\sim}$ defined below is a Pre A*-algebra.

∧	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

∨	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^{\sim}$
0	1
1	0
2	2

Note: The elements 0, 1, 2 in the above example satisfy the following laws:

(a) $2^{\sim} = 2$ (b) $1 \wedge x = x$ for all $x \in \mathbf{3}$

(c) $0 \vee x = x$ for all $x \in \mathbf{3}$ (d) $2 \wedge x = 2 \vee x = 2$ for all $x \in \mathbf{3}$.

Example: **2** = {0, 1} with operations $\wedge$, $\vee$, $(-)^{\sim}$ defined below is a Pre A*-algebra.

∧	0	1
0	0	0
1	0	1

∨	0	1
0	0	1
1	1	1

x	$x^{\sim}$
0	1
1	0

Note:

(i) $\left(2, \vee, \wedge, (\tilde{-})\right)$ is a Boolean algebra. So every Boolean algebra is a Pre A* algebra.

(ii) The identities 1.2(a) and 1.2(d) imply that the varieties of Pre A*-algebras satisfies all the dual statements of 1.2(b) to 1.2(g).

Definition: Let A be a Pre A*-algebra. An element $x \in A$ is called a central element of A if $x \vee x^{\sim} = 1$ and the set $\{x \in A / x \vee x^{\sim} = 1\}$ of all central elements of A is called the centre of A and it is denoted by B (A).

Theorem: [Satyanarayana.A, (2012)]: Let A be a Pre A*-algebra with 1, then B (A) is a Boolean algebra with the induced operations $\wedge$, $\vee$, $(-)^{\sim}$

Lemma: [Satyanarayana.A, (2012)]: Every Pre A*-algebra with 1 satisfies the following laws

(a) $x \vee 1 = x \vee x^{\sim}$ (b) $x \wedge 0 = x \wedge x^{\sim}$

Lemma: [Satyanarayana.A, (2012)]: Every Pre A*-algebra with 1 satisfies the following laws.

(a) $x \wedge (x^{\sim} \vee x) = x \vee (x^{\sim} \wedge x) = x$ (b) $(x \vee x^{\sim}) \wedge y = (x \wedge y) \vee (x^{\sim} \wedge y)$

(c) $(x \vee y) \wedge z = (x \wedge z) \vee (x^{\sim} \wedge y \wedge z)$

Lattice Structure on Pre A*-algebra: In this section we define (for any subset L of a Pre A*-algebra) a Pre A*-lattice (L, ∧, ∨). We give some examples of Pre A*-lattices. We give some properties of Pre A*-lattices. We define the congruence $\theta_a = \{(x, y) \in L \times L \,/\, a \wedge x = a \wedge y\}$ for any $a \in L$ and studied their properties. We also introduce the concept of Ideal, Ideal congruences on Pre A*-lattice and derived some important properties of these.

Definition: Let A be a Pre A*-algebra. A non-empty subset L of a Pre A*-algebra A in which for each pair of elements a ∈ A, b ∈ B(A) in L has greatest lower bound a ∨ b exists in L. Such a defined set L in Pre A*-algebra is said to be Pre A*-lattice.

Now we give another type of definition other than that in 2.1. Definition by means of equations.

Definition: Let A be a Pre A*-algebra. A non-empty subset L of a Pre A*-algebra A, equipped with two binary operations meet (∧) and join (∨) which assign to every pair a ∈A, b ∈B(A) of the elements of L, uniquely an element

a ∧ b as well as element a ∨ b in L in such a way that the following axioms holds.

(i) a ∧ (b ∧ c) = (a ∧ b) ∧ c, ∀ a, b, c ∈L (associative)

(ii) a ∧ b = b ∧ a, ∀ a, b ∈ L (commutative);

(iii) a ∧ (a ∨ b) = a, ∀ a, b ∈ L (absorption law)

2.1. Note: The above axioms (i), (ii), (iii) holds with respect to ⍰ also.

2.1. Example:

1. Let A be a Pre A*-algebra and **2** = {0,1} is a subset of A then **2** = {0,1} is a Pre A*-lattice.
2. **3** = {0, 1, 2} is a subset of a Pre A*-algebra then **3** = {0, 1, 2} is a Pre A*-lattice.
3. Fig (i) is an example of Pre A*-lattice.

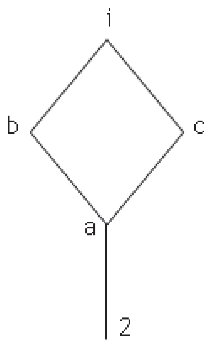

Figure (i)

4. Example of a poset which is shown in Fig (ii) is not a Pre A*-lattice:

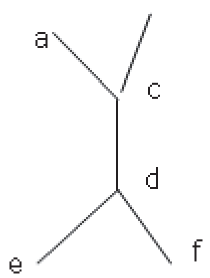

Figure (ii)

Definition: Let L_1 and L_2 be a two Pre A*-lattices. A mapping $f : L_1 \to L_2$ is called a homomorphism if

(i) $f(a \wedge b) = f(a) \wedge f(b)$ (ii) $f(a \vee b) = f(a) \vee f(b)$

The homomorphism $f : L_1 \to L_2$ is onto, then f is called epimorphism.

The homomorphism $f : L_1 \to L_2$ is one-one then f is called monomorphism

The homomorphisn $f: L_1 \to L_2$ is one-one and onto then f is called an isomorphism, and L_1, L_2 are isomorphic, denoted in symbol $L_1 \cong L_2$.

Theorem: Let A be a Pre A*-algebra. L is a subset of A Then (L, $\wedge$, $\vee$) is a Pre A*-lattice.

Proof: Since A is a Pre A*-algebra L is a subset of A.

We have a $\wedge$ (b $\wedge$ c) = (a $\wedge$ b) $\wedge$ c, $\forall$ a, b, c $\in$ A by 1.2 (e)

a $\wedge$ b = b $\wedge$ a, $\forall$ a, b $\in$ A by 1.2 (c) and a $\wedge$ a = a, $\forall$ a $\in$ A by 1.2 (b)

Therefore a $\wedge$ (b $\wedge$ c) = (a $\wedge$ b) $\wedge$ c, $\forall$ a, b, c $\in$ L, a $\wedge$ b = b $\wedge$ a, $\forall$ a, b $\in$ L

And a $\wedge$ a = a, $\forall$ a $\in$ L. Hence (L, $\wedge$) is a semilattice.

Similarly we can prove (L, $\vee$) is a semilattice. And since in a Pre A*-algebra,

a $\wedge$ (a $\vee$ b) = a, $\forall$ a $\in$ A, b $\in$ B(A). So, a $\wedge$ (a $\vee$ b) = a, $\forall$ a, b $\in$ L (absorption law). Hence (L, $\wedge$, $\vee$) is a Pre A*-lattice.

Definition: Let A be a Pre A*-algebra suppose L_1 be a subset of a Pre A*-lattice L. We say L_1 is a sub Pre A*-lattice of L if L_1 itself is a Pre A*-lattice (with respect to the operations of L).

Example: **3** = {0, 1, 2} is a Pre A*-lattice then **2** = {0,1} which is a subset of **3** = {0, 1, 2} is a sub Pre A*-lattice.

Definition: Let A be a Pre A*-algebra and L is subset of A. Then a Pre A*-lattice (L, $\wedge$, $\vee$) is said to be distributive Pre A*-lattice if any elements a, b, c in L we have the distributive law.

a $\wedge$ (b $\vee$ c) = (a $\wedge$ b) $\vee$ (a $\wedge$ c), $\forall$ a, b, c $\in$ L

Example: The chain shown in the Fig (iii) of section 2 is a distributive Pre A*-lattice.

Theorem: Let A be a Pre A*-algebra and L is a subset of A which is Pre A*-lattice. Then L becomes a distributive Pre A*-Lattice.

Proof: Since A is a Pre A*-algebra, and L is a subset of A which is clearly a Pre A*-lattice and since distributive law holds in a Pre A*-algebra,

a $\wedge$ (b $\vee$ c) = (a $\wedge$ b) $\vee$ (a $\wedge$ c), $\forall$ a, b, c $\in$ L.Hence L becomes a distributive Pre A*-lattice.

Definition: Let A be a Pre A*-algebra and L is a distributive Pre A*-Lattice. Define for any $a \in L$, $\theta_a = \{(x, y) \in L \times L \,/\, a \wedge x = a \wedge y\}$

Lemma: Let L be a distributive Pre A*-Lattice. Then $\theta_a = \{(x, y) \in L \times L \,/\, a \wedge x = a \wedge y\}$ is a congruence relation on L.

Proof: Since $a \wedge x = a \wedge x$ then $(x, x) \in \theta_a$, the relation is reflexive.

Let $(x, y) \in \theta_a$ then $a \wedge x = a \wedge y$

$\Rightarrow a \wedge y = a \wedge x$

$\Rightarrow (y, x) \in \theta_a$, the relation is symmetric

Let $(x, y) \in \theta_a$ and $(y, z) \in \theta_a$ then $a \wedge x = a \wedge y$ and $a \wedge y = a \wedge z$

$\Rightarrow a \wedge x = a \wedge z$

$\Rightarrow (x, z) \in \theta_a$, the relation is transitive.

Hence the relation θ_a is equivalence relation.

Let $x, y, z, t \in L$ such that $(x, y) \in \theta_a$ and $(z, t) \in \theta_a$ then $a \wedge x = a \wedge y$, $a \wedge z = a \wedge t$

Now $a \wedge (x \vee z) = (a \wedge x) \vee (a \wedge z)$

$= (a \wedge y) \vee (a \wedge t)$

$= a \wedge (y \vee t)$

This shows that $(x \vee z, y \vee t) \in \theta_a$

Hence θ_a is closed under $\vee$.

Also $a \wedge (x \wedge z) = (a \wedge x) \wedge (a \wedge z)$

$= (a \wedge y) \wedge (a \wedge t)$

$= a \wedge (y \wedge t)$

This shows that $(x \wedge z, y \wedge t) \in \theta_a$

Hence θ_a is closed under $\wedge$.

Therefore θ_a is a congruence relation on L.

Theorem: Let L be a distributive Pre A*-Lattice. Define $X = \{\theta_a / a \in L\}$ then $(X, \subseteq)$ is a distributive Pre A*-Lattice.

Proof: Clearly $\theta_a \subseteq \theta_a$ for all $\theta_a \in X$, shows $\subseteq$ is reflexive.

Let $\theta_a \subseteq \theta_b$ and $\theta_b \subseteq \theta_a$ implies that $\theta_a = \theta_b$.This $\subseteq$ is anti- symmetric.

Let $\theta_a \subseteq \theta_b$ and $\theta_b \subseteq \theta_c$ then $\theta_a \subseteq \theta_c$, shows that $\subseteq$is transitive.

Hence $(X, \subseteq)$is a poset.

First we show that $\theta_a \cap \theta_b = \theta_{a\vee \mathrm{b}}$

Let $(x, y) \in \theta_a \cap \theta_b$ then $a \wedge x = a \wedge y$ and $b \wedge x = b \wedge y$

Now $(a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x)$

$= (a \wedge y) \vee (b \wedge y)$

$= (a \vee b) \wedge y$

This implies that $(x, y) \in \theta_{a\vee b}$

This shows that $\theta_a \cap \theta_b \subseteq \theta_{a\vee \mathrm{b}}$.

On the other hand let $(x, y) \in \theta_{a\vee b}$, then $(a \vee b) \wedge x = (a \vee b) \wedge y$.

Now $a \wedge x = (a \wedge (a \vee b)) \wedge x$ (by absorption law)

$= a \wedge ((a \vee b) \wedge x)$

$= a \wedge ((a \vee b) \wedge y)$

$= (a \wedge (a \vee b)) \wedge y$

$= a \wedge y$

This implies that $(x, y) \in \theta_a$, hence $\theta_{a\vee \mathrm{b}} \subseteq \theta_a$

Similarly we can prove that $\theta_{a\vee \mathrm{b}} \subseteq \theta_b$

This implies that $\theta_{a\vee \mathrm{b}} \subseteq \theta_a \cap \theta_b$

Hence $\theta_a \cap \theta_b = \theta_{a\vee \mathrm{b}}$

Let $(r, s) \in \theta_a$ then $a \wedge r = a \wedge s \Rightarrow a \wedge b \wedge r = a \wedge b \wedge s \Rightarrow (r, s) \in \theta_{a\wedge b}$.

Hence $\theta_a \subseteq \theta_{a\wedge \mathrm{b}}$. Similarly $\theta_b \subseteq \theta_{a\wedge \mathrm{b}}$.

Thus $\theta_{a\wedge \mathrm{b}}$ is an upper bound of {θ_a, θ_b}.

Let θ_c is an upper bound of {θ_a, θ_b}.

Let $(x, y) \in \theta_{a\wedge b}$ then $a \wedge b \wedge x = a \wedge b \wedge y \Rightarrow (b \wedge x, b \wedge y) \in \theta_a \subseteq \theta_c$

$\Rightarrow c \wedge b \wedge x = c \wedge b \wedge y \Rightarrow b \wedge c \wedge x = b \wedge c \wedge y \Rightarrow (c \wedge x, c \wedge y) \in \theta_b \subseteq \theta_c$

$\Rightarrow c \wedge c \wedge x = c \wedge c \wedge y \Rightarrow c \wedge x = c \wedge y \Rightarrow (x, y) \in \theta_c$

Hence $\theta_{a\wedge \mathrm{b}} \subseteq \theta_c$.

Therefore Sup{θ_a, θ_b} = $\theta_{a\wedge \mathrm{b}}$ i.e., $\theta_a \vee \theta_b = \theta_{a\vee \mathrm{b}}$.

Hence X is a Pre A*-Lattice.

Since a $\wedge$ (b $\vee$ c) = (a $\wedge$ b) $\vee$ (a $\wedge$ c), $\forall$ a, b, c $\in$ L we have X is a distributive Pre A*-Lattice.

Definition: A nonempty subset I of a distributive Pre A*-Lattice L is said to be an ideal of L if the following hold.

(i) a ,b $\in$ I $\Rightarrow$ a $\vee$ b $\in$ I

(ii) a $\in$ I $\Rightarrow$ x $\wedge$ a $\in$ I for each x $\in$ L

Theorem: Let L be a distributive Pre A*-Lattice. Then F(L) the set of all ideals of L is a distributive Pre A*-Lattice under the set inclusion.

Proof: Let I, J $\in$ F(L)

Clearly I $\bigcap$ J is an ideal of L, and I $\bigcap$ J = Inf{I, J} in the poset (F(L), $\subseteq$).

Let H = { $\bigvee_{i=1}^{n} x_i$ / $x_i \in I \bigcup J$, n is a positive integer }

Let x, y $\in$ H implies that x = $\bigvee_{i=1}^{n} x_i$, y = $\bigvee_{i=1}^{n} y_i$ and hence $x \vee y = \bigvee_{k=1}^{n} t_k$

(each $t_i \in I \bigcup J$)

Let a $\in L$ and x $\in$ H.

Then $a \wedge x = a \wedge (\bigvee_{i=1}^{n} x_i) = \bigvee_{i=1}^{n} (a \wedge x_i)$

Now $a \wedge x_i \in I \bigcup J$ (since if $x_i \in I$ then $a \wedge x_i \in I$ and if $x_i \in J$ then $a \wedge x_i \in J$)

Hence $a \wedge x \in H$.

Therefore H is an ideal of L.

Clearly I, J are subsets of H.

Let K be any ideal of L such that $I \subseteq K$, $J \subseteq K$.

Now let $x \in H \Rightarrow x = \bigvee_{i=1}^{n} x_i \Rightarrow x \in K$(since $I, J \subseteq K$ and K is an ideal)

Hence $H \subseteq K$

Therefore H is the smallest ideal containing I, J.

Therefore Sup{I, J} = H.i.e., $I \vee J$ = { $\bigvee_{i=1}^{n} x_i$ / $x_i \in I \bigcup J$, n is a positive integer }

Hence $F(L)$is a lattice under the set inclusion.

Let I, J, K $\in F(L)$

Clearly $(I \bigcap J) \vee (I \bigcap K) \subseteq I \bigcap (J \vee K)$

Let $t \in I \bigcap (J \vee K)$

Then $t = \bigvee_{i=1}^{n} x_i$, where $x_i \in J \bigcup K$

Now $t = t \wedge t = t \wedge (\bigvee_{i=1}^{n} x_i) = \bigvee_{i=1}^{n} (t \wedge x_i)$

Now $t \wedge x_i \in I \bigcap J$ or $I \bigcap K$

Therefore $t \in (I \bigcap J) \vee (I \bigcap K)$

Hence $I \bigcap (J \vee K) \subseteq (I \bigcap J) \vee (I \bigcap K)$

Thus $I \bigcap (J \vee K) = (I \bigcap J) \vee (I \bigcap K)$

Therefore $F(L)$ is a distributive Pre A*-Lattice.

Definition: For any ideal I of a Pre A*-Lattice L we define

θ_I = {(x , y) / $a \wedge x = a \wedge y$, for some $a \in I$}. That is $\theta_I = \bigcup_{a \in I} \theta_a$

Theorem: θ_I is a congruence on a Pre A*-Lattice L for any ideal I of L.

Proof: We know that the union of a class of congruences on L is again a congruence on L if the given class is directed above, in the sense that, for any two members θ_1 and θ_2 in that class there exist a member θ in the class containing both θ_1 and θ_2.

Now consider C = { θ_a / $a \in I$}

Since each θ_a is congruence on L, C is a class of congruence on L. Also for any a, b $\in$ I we have $a \vee b \in I$ and $\theta_a \vee \theta_b = \theta_{a \vee b} \in C$

Therefore C is a directed above class of congruences and $\bigcup_{a \in I} \theta_a$ ($= \theta_I$) is a congruence on L.

Theorem: For any ideals I and J of a Pre A*-Lattice L the following hold.

(1) $I \subseteq J \Rightarrow \theta_I \subseteq \theta_J$

(2) $\theta_I \cap \theta_J = \theta_{I\cap J}$

(3) $\theta_I \vee \theta_J = \theta_{I\vee J}$

Proof: Let I and J are ideals of a Pre A*-Lattice L.

(1) Suppose that $I \subseteq J$.

Let $a \in I \Rightarrow a \in J$

Let $(x, y) \in \theta_I \Rightarrow a \wedge x = a \wedge y$ for some $a \in I$

$\Rightarrow a \wedge x = a \wedge y$ for some $a \in J$

$\Rightarrow (x, y) \in \theta_J$

Therefore $\theta_I \subseteq \theta_J$.

(2) Since $I \cap J \subseteq I$ and $I \cap J \subseteq J$ then by (1) we get $\theta_{I\cap J} \subseteq \theta_I$ and $\theta_{I\cap J} \subseteq \theta_J$

Hence $\theta_{I\cap J} \subseteq \theta_I \cap \theta_J$

Let $(x, y) \in \theta_I \cap \theta_J \Rightarrow (x, y) \in \theta_I$ and $(x, y) \in \theta_J$

$\Rightarrow a \wedge x = a \wedge y$ and $b \wedge x = b \wedge y$, where $a \in I$, $b \in J$

Now $a \wedge b \in I \cap J$ and also $(a \wedge b) \wedge x = a \wedge (b \wedge x)$

$= a \wedge (b \wedge y)$

$= (a \wedge b) \wedge y$

Therefore $(x, y) \in \theta_{I\cap J}$

Hence $\theta_I \cap \theta_J \subseteq \theta_{I\cap J}$

Therefore $\theta_I \cap \theta_J = \theta_{I\cap J}$

(3) Since $I \subseteq I \vee J$ and $J \subseteq I \vee J$ then by (1) we get $\theta_I \subseteq \theta_{I\vee J}$, $\theta_J \subseteq \theta_{I\vee J}$ and hence

$\theta_I \vee \theta_J \subseteq \theta_{I\vee J}$

Let $(x, y) \in \theta_z$ where $z \in I \vee J$

$\Rightarrow z = \bigvee_{i=1}^{n} x_i$ for some $x_i \in I \vee J$ and $(x, y) \in \theta_{\vee_{1=1}^{n} x_i} = \bigvee_{i=1}^{n} \theta_{x_i}$ (since $\theta_{a\vee b} = \theta_a \vee \theta_b$)

$\Rightarrow (x, y) \in \bigvee_{i=1}^{n} \theta_{x_i} \subseteq \theta_I \vee \theta_J$ (since each $x_i \in I$ or J)

$\Rightarrow (x, y) \in \theta_I \vee \theta_J$

$\Rightarrow \theta_{I\vee J} \subseteq \theta_I \vee \theta_J$

Therefore $\theta_I \vee \theta_J = \theta_{I\vee J}$.

Let us recall that the set Con(L) of all congruences on any algebra L is an algebraic lattice under the inclusion ordering in which the g.l.b and l.u.b of any subset Č of Con(L) are given by g.l.b Č = $\bigcap_{\theta \in \check{C}} \theta$ and l.u.b Č = $\bigcup \{\theta_1 \, o \, \theta_2 \, o \ldots o \, \theta_n / \theta_i \in$ Č }.

Now we have the following.

Theorem: Let F(L) be the lattice of all ideals a Pre A*-Lattice L. Then $I \rightarrow \theta_I$ is homomorphism of the lattice F(L) into the lattice Con(L) of all congruences on L.

Proof: From 2.6. Theorem (2 and 3) it follows that $I \rightarrow \theta_I$ is lattice homomorphism of F(L) into the lattice Con(L).

Lemma: Let L be a distributive Pre A*-Lattice. Then $L_a = \{ a \wedge x / x \in L\}$ is a sub algebra of L and it is a distributive lattice.

Proof:

Let $a \wedge x$, $a \wedge y \in L_a$

Then $a \wedge (x \wedge y) = (a \wedge x) \wedge (a \wedge y) \in L_a$

Hence L_a is closed under $\wedge$

Also $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) \in L_a$

Hence L_a is closed under $\vee$

Therefore L_a is a sub algebra of L. Since L is a distributive Pre A*-Lattice we have L_a is a distributive lattice.

Theorem: Let L be a distributive Pre A*-Lattice. Then the map $f_a : L \to L_a$ defined by $f_a(x) = a \wedge x$ is a homomorphism and $L/\theta_a \cong L_a$.

Proof: Let $x, y \in L$. Then

$f_a(x \vee y) = a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) = f_a(x) \vee f_a(y)$ and

$f_a(x \wedge y) = a \wedge (x \wedge y) = (a \wedge x) \wedge (a \wedge y) = f_a(x) \wedge f_a(y)$

Therefore f_a is homomorphism.

For $a \wedge x \in L_a$, $f_a(x) = a \wedge x$

Hence f_a is onto.

Now ker $f_a = \{(x, y) / f_a(x) = f_a(y)\} = \{(x, y) / a \wedge x = a \wedge y\} = \theta_a$

By fundamental theorem of homomorphism $L/\ker f_a \cong L_a$, which imply $L/\theta_a \cong L_a$.

Conclusion: This manuscript acknowledged the construction of lattice structure on Pre A*-algebra and such a defined lattice structure on Pre A*-algebra was referred as Pre A*-lattice L. Further various properties of the Pre A*-lattice were obtained. Also attained to construct the congruence $\theta_a = \{(x, y) \in L \times L / \ a \wedge x = a \wedge y\}$ for any $a \in L$ and $X = \{\theta_a / a \in L\}$ then $(X, \subseteq)$ is a distributive Pre A*-Lattice. Also it has been defined an ideal on Pre A*-lattice L and proved that F (L), the set of all ideals of L is a distributive Pre A*-lattice under the set inclusion. For any ideal I of a Pre A*-Lattice L it is defined $\theta_I = \{(x, y) / a \wedge x = a \wedge y$, for some $a \in I\}$ is a congruence on a Pre A*-Lattice and confirmed that $I \to \theta_I$ is homomorphism of the lattice F (L) into the lattice Con(L) of all congruences on L. Finally it has been identified a distributive lattice $L_a = \{ a \wedge x / x \in L\}$ and the map $f_a : L \to L_a$ defined by $f_a(x) = a \wedge x$ is a homomorphism and $L/\theta_a \cong L_a$.

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