INVERSE LIMITS OF A - ALGEBRAS*

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Abstract: The concept of A*-algebra is firstly introduced by Koteswara Rao. P in his thesis. For X, $\mathbf{3}^X$ is an A*-algebra, where $\mathbf{3}$ = {0,1,2}. Every A*-algebra can be imbedded in $\mathbf{3}^X$ for some X. In this paper, we present the concept of A*-algebras, their properties, examples of A*-algebras and we introduce inverse system of A*-algebras, dual target, inverse limits of inverse system of A*-algebras and their existence and uniqueness.

Keywords: A* - Algebra, Inverse System, Dual Target, Inverse Limit For The Inverse System.

Introduction:

Definition: An algebra $(A, \wedge, *, (-)^{\sim}, (-)_\pi, 1)$ is an A*-algebra if it satisfies:

(i) $a_\pi \vee (a_\pi)^{\sim}=1$, $(a_\pi)_\pi = a_\pi$ where $a \vee b = (a^{\sim} \wedge b^{\sim})^{\sim}$

(ii) $a_\pi \vee b_\pi = b_\pi \vee a_\pi$

(iii) $(a_\pi \vee b_\pi) \vee c_\pi = a_\pi \vee (b_\pi \vee c_\pi)$

(iv) $(a_\pi \wedge b_\pi) \vee (a_\pi \wedge (b_\pi)^{\sim}) = a_\pi$

(v) $(a \wedge b)_\pi = a_\pi \wedge b_\pi$, $(a \wedge b)^{\#} = a^{\#} \vee b^{\#}$ where $a^{\#} = (a_\pi \vee a^{\sim}{}_\pi)^{\sim}$

(vi) $a^{\sim}{}_\pi = (a_\pi \vee a^{\#})^{\sim}$, $a^{\sim\#} = a^{\#}$

(vii) $(a * b)_\pi = a_\pi$, $(a * b)^{\#} = (a_\pi)^{\sim} \wedge (b^{\sim}{}_\pi)^{\sim}$

(viii) $a = b$ if and only if $a_\pi = b_\pi$, $a^{\#} = b^{\#}$.

We write 0 for $1^{\sim}$, 2 for $0 * 1$.

Example: $\mathbf{3}$ = {0, 1, 2} with the operations defined below is an A*- algebra.

∧	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

∨	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

*	0	1	2
0	0	2	2
1	1	1	1
2	0	2	2

x	0	1	2
x	1	0	2
x_π	0	1	0
$x^{\#}$	0	0	1

Note: From 1(i) through 1(iv) and Huntington's Theorem B (A) = {a_π | a ∈A} is a Boolean algebra with ∧, ∨, 〚(-)〛 ^∼, 0 and a ∈ B (A) ⇒a_π = a.

Since 1, 0, (a_π)^∼∈ B (A), we have 1_π=1, 0_π=0, (a_π)_(π)^∼=(a_π)^∼ and a_π∧a^#=0, a *0=a_π .

Lemma: For any x, y, z in an A*- algebra

(i) $x^{\sim\sim} = x$

(ii) $(x \wedge y)^{\sim}{}_\pi = (x^{\sim} \wedge y)_\pi \vee (x \wedge y^{\sim})_\pi \vee (x^{\sim} \wedge y^{\sim})_\pi$

(iii) $(x \vee y)_\pi = (x^{\sim} \wedge y)_\pi \vee (x \wedge y^{\sim})_\pi \vee (x \wedge y)_\pi$

(iv) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Lemma: For any x,y in A

(i) $(x * y)^{\sim}{}_\pi = (x_\pi)^{\sim} \wedge (y^{\sim})_\pi$

(ii) $x = x_\pi * (x^{\sim})_\pi{}^{\sim} = (x_\pi) * x^{\#}$

(iii) If $x = e * f$, where $e, f \in B(A)$, $e \wedge f = 0$, then $x_\pi = e$, $x^{\#} = f$.

Theorem: Every A*-algebra $(A, \wedge, *, (-)_\pi, (-)^{\sim}, 1)$ satisfies the following conditions:

For x, y, z in A

(i) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

(ii) $x \wedge y = y \wedge x$

(iii) $x \wedge x = x$

(iv) $1 \wedge x = x$

(v) $x^{\sim\sim} = x$

(vi) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ where $x \vee y = (x^{\sim} \wedge y^{\sim})^{\sim}$

(vii) $1_\pi = 1$

(viii) $[(x_\pi)^{\sim}]_\pi = (x_\pi)^{\sim}$

(ix) $(x \wedge y)_\pi = x_\pi \wedge y_\pi$

(x) $(x \wedge x^{\sim})_\pi = 0$ where $1^{\sim}=0$

(xi) $x_\pi \wedge (x_\pi \vee y_\pi) = x_\pi$
(xii) $(x \wedge y)^{\sim}_{\pi} = (x \wedge y^{\sim})_\pi \vee (x^{\sim} \wedge y)_\pi \vee (x^{\sim} \wedge y^{\sim})_\pi$
(xiii) $(x_\pi)_\pi = x_\pi$
(xiv) $(x * y)_\pi = x_\pi$
(xv) $(x * y)^{\sim}_{\pi} = (x_\pi)^{\sim} \wedge (y^{\sim})_\pi$
(xvi) $x = x_\pi * (x^{\sim}_{\pi})^{\sim}$

Theorem: An algebra $(A, \wedge, *, (-)^{\sim}, (-)_\pi, 1)$ satisfying axioms of the above theorem is an A*-algebra.

Proposition: In an A*- algebra A, if $0 \neq 1$, then A has atleast 3 elements.

Proposition: If $(A, \wedge, *, (-)^{\sim}, (-)_\pi, 1)$ is an A*- algebra and $a, b \in A$, then

(i) $a_\pi = a$ if and only if $a^{\#} = 0$
(ii) $a^{\#}{}_{\pi} = a^{\#}$
(iii) $a^{\#\sim}{}_{\pi} = a^{\#\sim}$
(iv) $a^{\#\#} = 0$
(v) $a_\pi \wedge a^{\#\sim} = a_\pi$
(vi) $a^{\#\sim} \wedge a^{\sim}{}_{\pi} = a^{\sim}{}_{\pi}$
(vii) $(a^{\#}) \wedge a^{\sim}{}_{\pi} = a^{\#} \wedge a_\pi = 0$
(viii) $(a_\pi)^{\sim} \wedge a^{\#} = a^{\#}$
(ix) $a^{\sim}{}_{\pi} \wedge (a_\pi)^{\sim} = a^{\sim}{}_{\pi}$

Example: Let X be a nonempty set.

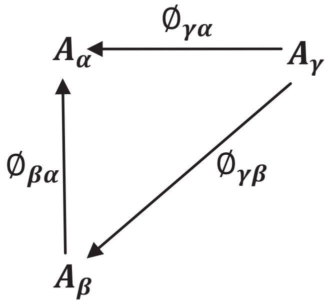
$T_X = \{(P, F) : P \subseteq X,\ F \subseteq X,\ P \cap F = \phi\}$. For $A, B \subseteq X$ denote intersection, union and complementation in the Boolean algebra P (X) by AB, A+B, A'. Define $\wedge, \vee, (-)^{\sim}, (-)_\pi, 1$ on T_X by
∨, 〖(-)^~,(-)〗_π,1 on T_X by

(i) (P, F) ∧ (P1, F1) = (PP1 , PF1+FF1+FP1)
(ii) (P, F) ∨ (P1, F1) = (PF1+PP1+FP1 , FF1)
(iii) (P, F)^~= (F, P)
(iv) 1 = (X, ϕ)
(v) (P, F)_π= (P, P^')
(vi) (P, F) * (P1, F1) = (P, P^'F1)

Then (T_X, ∧, 〖〖(-)〗^~,(-)〗_π,*,1) is an A*-algebra.

Definition: An inverse system of A*-algebras is a family $\{A_\alpha\}_{\alpha \in I}$, where I is a preordered set, of A*-algebras together with, for each α, β with $\alpha \leq \beta$, there is an A*-homomorphism $\emptyset_{\beta\alpha}: A_\beta \to A_\alpha$ such that

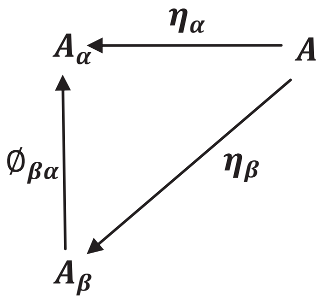
(1) $\emptyset_{\gamma\alpha} = \emptyset_{\beta\alpha}\emptyset_{\gamma\beta}$ if $\alpha \leq \beta \leq \gamma$ i.e., the diagram



is commutative.

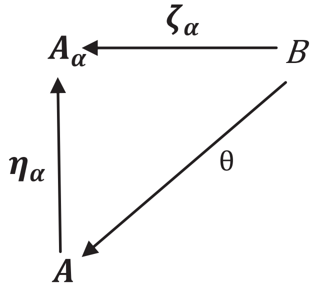
(2) $\emptyset_{\alpha\alpha} = 1_{A_\alpha}$.

Definition: Given an inverse system $(A_\alpha, \emptyset_{\beta\alpha})_{\alpha \in I}$ with $\alpha \leq \beta$ of A*-algebras. A dual target for the system is a set $(A, \eta_\alpha)_{\alpha \in I}$, where A is an A*-algebra and $\eta_\alpha : A \to A_\alpha$ such that $\emptyset_{\beta\alpha}\eta_\beta = \eta_\alpha$ for $\alpha \leq \beta$ i.e., the diagram



is commutative.

Definition: An inverse limit for the inverse system $(A_\alpha, \emptyset_{\beta\alpha})_{\alpha\in I}$ of A*-algebras is a dual target $(A, \eta_\alpha)_{\alpha\in I}$ satisfying the universal property:
For any dual target $(B, \zeta_\alpha)_{\alpha\in I}$ then there exists a unique A*-homomorphism $\theta : B \to A$ such that $\eta_\alpha\theta = \zeta_\alpha$, $\alpha \in I$. We denote the inverse limit $(A, \eta_\alpha)_{\alpha\in I}$ by $\overset{\lim}{\longleftarrow} (A_\alpha, \emptyset_{\beta\alpha})$.



is commutative

i.e., $\boldsymbol{\eta_\alpha\theta = \zeta_\alpha, \alpha \in I}$.

Note: We have the usual uniqueness upto isomorphism:
If $(A', \{\eta_{\alpha'}\})$ is a second inverse limit, then there exists a unique isomorphism $\theta : A' \to A$ such that $\eta_\alpha\theta = \eta_{\alpha'}$.
Theorem: Every inverse system of A*-algebras has an inverse limit.
Proof: Suppose $\{A_\alpha, \emptyset_{\beta\alpha}\}_{\alpha\in I}$, $\alpha \le \beta$ is an inverse system of A*-algebras.
Define $A = \{a \in \prod_{\alpha\in I} A_\alpha \mid p_\alpha(a) = \emptyset_{\beta\alpha}\, p_\beta(a), \alpha \le \beta\}$.
Claim: A is a sub algebra of $\prod_{\alpha\in I} A_\alpha$.
Consider $p_\alpha(0) = 0_\alpha = \emptyset_{\beta\alpha}(0_\beta) = \emptyset_{\beta\alpha} p_\beta(0)$ where $o \in \prod_{\alpha\in I} A_\alpha$.
Therefore $o \in A$.
Therefore $A \neq \phi$.
Similarly 1, 2 in $\prod_{\alpha\in I} A_\alpha$ are in A.
Let $a^{(1)}, a^{(2)} \in A \Rightarrow a^{(1)}, a^{(2)} \in \prod_{\alpha\in I} A_\alpha$
and $p_\alpha(a^{(1)}) = \emptyset_{\beta\alpha} p_\beta(a^{(1)})$ and
$p_\alpha(a^{(2)}) = \emptyset_{\beta\alpha} p_\beta(a^{(2)})$.
$p_\alpha(a^{(1)} \wedge a^{(2)}) = p_\alpha(a^{(1)}) \wedge p_\alpha(a^{(2)})$
$= \emptyset_{\beta\alpha} p_\beta(a^{(1)}) \wedge \emptyset_{\beta\alpha} p_\beta(a^{(2)})$
$= \emptyset_{\beta\alpha} p_\beta(a^{(1)} \wedge a^{(2)})$.
Therefore $a^{(1)} \wedge a^{(2)} \in A$.
Similarly $a^{(1)} * a^{(2)} \in A$ and $a_\pi, a^\sim \in A$ for every $a \in A$.
Therefore A is a sub A*-algebra of $\prod_{\alpha\in I} A_\alpha$.
Define $\eta_\alpha = p_\alpha / A$.
Clearly $\eta_\alpha = \emptyset_{\beta\alpha}\eta_\beta$, $\alpha \le \beta$.
Therefore $(A, \eta_\alpha)_{\alpha\in I}$ is a dual target of $\{A_\alpha, \emptyset_{\beta\alpha}\}_{\alpha\in I}$, $\alpha \le \beta$.
Suppose $(B, \zeta_\alpha)_{\alpha\in I}$ is another dual target of $\{A_\alpha, \emptyset_{\beta\alpha}\}_{\alpha\in I}$
i.e., $\zeta_\alpha = \emptyset_{\beta\alpha}\zeta_\beta$.
Define $\theta' : B \to \prod_{\alpha\in I} A_\alpha$ by $\theta'(b) = a$, where $a_\alpha = \zeta_\alpha(b)$.
$p_\alpha(a) = a_\alpha = \zeta_\alpha(b) = \emptyset_{\beta\alpha}\zeta_\beta(b) = \emptyset_{\beta\alpha}(\zeta_\beta(b)) = \emptyset_{\beta\alpha}(a_\beta) = \emptyset_{\beta\alpha} p_\beta(a)$.
Therefore $a \in A$.
Therefore θ' defines a unique A*-homomorphism $\theta : B \to A$ such that $\theta(b) = \theta'(b)$.
Now $\eta_\alpha\, \theta(b) = \eta_\alpha(\theta(b)) = p_\alpha(a) = a_\alpha = \zeta_\alpha(b) = (\emptyset_{\beta\alpha} p_\beta)(b)$
$= \emptyset_{\beta\alpha}(p_\beta(b)) = \emptyset_{\beta\alpha}(\zeta_\beta(b)) = \emptyset_{\beta\alpha}(a_\beta) = a_\alpha = \zeta_\alpha(b)$.
Therefore $\eta_\alpha\, \theta = \zeta_\alpha$.
Therefore $(A, \eta_\alpha)_{\alpha\in I}$ is an inverse limit of the inverse system $(A_\alpha, \emptyset_{\beta\alpha})_{\alpha\in I}$.
Theorem: If $\{\emptyset_\alpha\}_{\alpha\in I}$ is a set of congruences on an A*-algebra B. If $I = \{\alpha\}$ is pre-ordered as $\alpha \le \beta$ if $\emptyset_\alpha \supset \emptyset_\beta$. Then $\{A_\alpha, \emptyset_{\beta\alpha}\}_{\alpha\in I}$, where $A_\alpha = B/\emptyset_\alpha$, has an inverse limit $\{A, \eta_\alpha\}_{\alpha\in I}$ and there is an A* - homomorphism $\theta : B \to A$ such that $\eta_\alpha\theta = \vartheta_\alpha$, where $\vartheta_\alpha : B \to A_\alpha$ is canonical epimorphism. If $\bigcap_{\alpha\in I} \emptyset_\alpha = 1_B$, then θ is a monomorphism.
Proof: Suppose B is an A*-algebra. Suppose $\{\emptyset_\alpha\}_{\alpha\in I}$ is a set of congruences on B. We preorder the set $I = \{\alpha\}$ by agreeing that $\alpha \le \beta$ if $\emptyset_\alpha \supset \emptyset_\beta$. Put $A_\alpha = B/\emptyset_\alpha$. For $\alpha \le \beta$ let $\emptyset_{\beta\alpha} : A_\beta \to A_\alpha$ is a homomorphism $\bar{b}_\beta \mapsto \bar{b}_\alpha$. Then $\emptyset_{\alpha\alpha} = 1_{A_\alpha}$, $\emptyset_{\beta\alpha}\emptyset_{\gamma\beta} = \emptyset_{\gamma\alpha}$ if $\alpha \le \beta \le \gamma$. Then $\{A_\alpha, \emptyset_{\beta\alpha}\}$ is an inverse system of algebras. Therefore the inverse limit $\{A, \eta_\alpha\}_{\alpha\in I}$ of $\{A_\alpha, \emptyset_{\beta\alpha}\}_{\alpha\in I}$ exists.

Define $\vartheta_\alpha : B \rightarrow A_\alpha$ by $\vartheta_\alpha (b) = \bar{b}_\alpha$. ϑ_α is canonical epimorphism for every $\alpha \in I$.
Clearly $\emptyset_{\beta\alpha}\vartheta_\beta = \vartheta_\alpha$ for $\alpha \leq \beta$. Therefore $\{B, \vartheta_\alpha\}_{\alpha\in I}$ is a dual target for $\{A_\alpha, \emptyset_{\beta\alpha}\}_{\alpha\in I}$.
Then there exists a homomorphism $\theta : B \rightarrow A$ such that $\eta_\alpha\theta = \vartheta_\alpha$.
If $\bigcap_{\alpha\in I} \emptyset_\alpha = 1_B$, then $\theta : B \rightarrow A$ is a homomorphism.
For, let $b, c \in B \ni \theta(b) = \theta(c)$

$\Rightarrow \eta_\alpha(\theta(b)) = \eta_\alpha(\theta(c)) \ \forall\ \alpha$
$\Rightarrow (\eta_\alpha\theta)(b) = (\eta_\alpha\theta)(c) \ \forall\ \alpha$
$\Rightarrow \vartheta_\alpha (b) = \vartheta_\alpha (c) \ \forall\ \alpha$
$\Rightarrow \bar{b}_\alpha = \bar{c}_\alpha \ \forall\ \alpha$
$\Rightarrow (b, c) \in \theta_\alpha \ \forall\ \alpha$
$\Rightarrow (b, c) \in \bigcap_{\alpha\in I} \emptyset_\alpha$
$\Rightarrow (b, c) \in 1_B$
$\Rightarrow b = c.$

Therefore $\theta(b) = \theta(c) \Rightarrow b = c$.
Therefore θ is a monomorphism.

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