# SOME RESULTS IN BITOPOLOGICAL SPACES

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**Abstract:** In this paper, we are going to introduce some properties of $\tau_1\tau_2 - \delta$ semi open sets / closed sets in bitopological spaces. In addition, we investigate several results in $\tau_1\tau_2 - \delta$ semi open sets / closed sets and $\tau_1\tau_2 - \delta$ semi continuous functions in bitopological spaces. Moreover, we show several results in $\tau_1\tau_2 - \delta$ semi open sets / closed sets in bitopological spaces. Bitopological space does not exist for every metric space. But it exists only for special type of metric spaces, known as "asymmetric metric spaces". There are many applications in different parts in mathematics.

**Keywords:** $\mathbf{T_1T_2 - \Delta}$ Semi Open Sets, $\mathbf{T_1T_2 - \Delta}$ Semi Closed Sets, $\mathbf{T_1T_2 - \Delta}$ Semi Continuous.

**Introduction:** J.C.Kelly started the study about bitopological spaces in 1963. He introduced the concept "bitopological space". Besides, he introduced various properties in bitopological spaces, and got some generalizations of some specific results. Kelly initiated his study about bitopological space from quesi-metric and its conjugate. A quasi-pseudo-metric $p(\ ,\ )$ on a set $X$ on the Cartesian product $X \times X$ satisfies the following properties: $p(x,x) = 0 , \forall\, x \in X$, $p(x,z) \leq p(x,y) + p(y,z)$, $\forall\, x,y,z \in X$ and $p(x,y) = 0$ iff $x = y , \forall\, x,y \in X$ , then $p(,)$ is a quasi-metric. However, the symmetric property does not hold for quasi-metric. Furthermore, every metric space is a quasi-metric space. But the converse need not be true. Bitopological spaces arise in a natural way by considering the topologies induced by sets of the form $B_{px\epsilon} = \{y : p(x,y) < \epsilon\}$ and $B_{qx\epsilon} = \{y : q(x,y) < \epsilon\}$ ; where $p$ and $q$ are quasi metrics on $X$ and $q(x,y) = p(y,x)$. For a nonempty set $X$, we define two topologies $\tau_1$ and $\tau_2$ on $X$. Then, $(X, \tau_1, \tau_2)$ is called a bitopological space. A topological space occurs for every metric space. But bitopological spaces occur for quasi metric spaces or asymmetric metric spaces. Quasi-uniform spaces, which are generalizations of quasi-metric spaces, also induce bitopological spaces. This structure is a richer structure than that of a topological space. Some authors extended the suitable generalizations of standard topological properties into bitopological category. Most of the results are related with the theory, but some with applications. Any subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called open, if $A$ is both $\tau_1 -$ open and $\tau_2 -$ open. Throughout this paper, $\tau_i - int(A)$, $\tau_i - cl(A), \tau_i - \delta int(A)$ and $\tau_i - \delta cl(A)$ be the interior, closure, $\delta -$interior and $\delta -$closure of $A$ with respect to the topology $\tau_i$ respectively, $i = 1,2$. Let $\tau_j - \delta int(A)$ and $\tau_j - \delta cl(A)$ are the $\delta -$interior and $\delta -$closure of $A$ with respect to the topology $\tau_j$ ; $j = 1s, 2s$. Semi open sets in bitopological spaces introduced by Maheswari and Prasad in 1977. Further properties were studied by Bose in 1981. Banerjee initiated the notion $\delta -$ open sets in bitopological spaces in 1987. Khedr introduced and studied about $\tau_1\tau_2 - \delta$ open sets. Later, Fukutake defined one kind of semi open sets and studied their properties in 1989. Any subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_{12} -$regular open, if $A = \tau_1 - int\big(\tau_2 - cl(A)\big)$. Any subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2 -$ semi open, if $A \subseteq \tau_2 - cl\big(\tau_1 - int(A)\big)$. In a bitopological space $(X, \tau_1, \tau_2)$, $A$ is said to be $\tau_1 - \delta$ open, if for $x \in A$, there exists $\tau_{12} -$regular open set $G$ such that $x \in G \subset A$. Complement of $\tau_1 - \delta$ open set is called $\tau_1 - \delta$ closed set. Collection of all $\tau_1 - \delta$ open sets and $\tau_2 - \delta$ open sets are denoted by $\tau_{1s}$ *and* $\tau_{2s}$ respectively. Always $\tau_{1s} \subset \tau_1$ and $\tau_{2s} \subset \tau_2$. Recently, Edward Samuel and Balan established $\tau_1\tau_2 - \delta$ semi open sets in bitopological spaces. Any subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2 - \delta$ semi open, if $U \subseteq A \subseteq \tau_2 - cl(U)$ , for some $\tau_1 - \delta$ open set $U$. Similarly, Any subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2 - \delta$ semi closed, if $F \supseteq A \supseteq \tau_2 - int(F)$ , for some $\tau_1 - \delta$ closed set $F$.

A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be pairwise continuous if and only if the induced functions $f : (X, \tau_1) \rightarrow (Y, \sigma_1)$ and $f : (X, \tau_2) \rightarrow (Y, \sigma_2)$ are continuous.
Consider the two bitopological spaces $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$. Then, a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called $\tau_1\tau_2 -\delta$ semi continuous, if $f^{-1}(V)$ is $\tau_1\tau_2 -\delta$ semi open set in $X$, for every $\sigma_1 -\delta$ open set $V$ in $Y$.

**Methodology:** We have introduced some definitions of various kind of open sets, semi open sets, closed sets, semi closed sets, continuity and semi continuity in bitopological spaces. In addition, we have discussed some properties of $\tau_1\tau_2 - \delta$ semi open/closed sets in bitopological spaces. We have proved the following : In a bitopological space $(X, \tau_1, \tau_2)$, $A$ is $\tau_1\tau_2 - \delta$ semi open iff $X\backslash A$ is $\tau_1\tau_2 - \delta$ semi closed.

Further, we have explained the properties of $\tau_1\tau_2 - \delta$ semi continuous functions in $(X, \tau_1, \tau_2)$. We have introduced the following results: Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ . Then, $f$ is $\tau_1\tau_2$–$\delta$ semi continuous iff $f^{-1}(U)$ is $\tau_1\tau_2$– $\delta$ semi closed in $X$, $\forall$ $\sigma_1$ –closed set $U$ in $Y$.

Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be $\tau_1\tau_2$–$\delta$ semi continuous. Then, $\forall$ $\sigma_1$–open set $V$ in $Y$, $\exists$ $\tau_1\tau_2$– $\delta$ semi open set $P$ in $X$ such that $f(P) \subseteq V$. Moreover, A constant function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a $\tau_1\tau_2$– $\delta$ semi continuous function. Besides, we have introduced the homeomorphism in bitopological spaces. With that, the following result is also proved: Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be bijective and homeomorphism. Then, $f$ is closed and continuous.

**Results and findings:** First we will show the following result : Let $A$ be a subset of a bitopological space $(X, \tau_1, \tau_2)$. Then, $A$ is $\tau_1\tau_2 - \delta$ semi open iff $X\backslash A$ is $\tau_1\tau_2 - \delta$ semi closed in $(X, \tau_1, \tau_2)$. To prove this result, let $A$ be $\tau_1\tau_2 - \delta$ semi open. Then, there exists a $\tau_1 - \delta$ openset $U$ such that $U \subseteq A \subseteq \tau_2 - cl(U)$. This implies, $\tau_2 - int(U^c) \subseteq A^c \subseteq U^c$. i.e. $\tau_2 - int(V) \subseteq X\backslash A \subseteq V$ ; where $U^c = V$ is a $\tau_1 - \delta$ closed set. Thus, $X\backslash A$ is $\tau_1\tau_2 - \delta$ semi closed. Conversely, let $X\backslash A$ is $\tau_1\tau_2 - \delta$ semi closed. Then, $\tau_2 - int(F) \subseteq X\backslash A \subseteq F$, for some $\tau_1 - \delta$ closed set $F$. This implies, $\tau_2 - cl(F^c) \supseteq A \supseteq F^c$ and $F^c$ is $\tau_1 - \delta$ open set. Thus, $A$ is $\tau_1\tau_2 - \delta$ semi open.

Every $\tau_1 - \delta$ open set is $\tau_1\tau_2 - \delta$ semi open in $(X, \tau_1, \tau_2)$. And every $\tau_1\tau_2 - \delta$ semi open set is $\tau_1\tau_2$ –semi open in $(X, \tau_1, \tau_2)$. Similary, we can prove the same results for closed sets also. However, the converse of the above statements need not be true.

If $A$ and $B$ are $\tau_1\tau_2 - \delta$ semi open sets in a bitopological space $(X, \tau_1, \tau_2)$, then $A \cup B$ is also $\tau_1\tau_2 - \delta$ semi open set. But $A \cap B$ may not be a $\tau_1\tau_2 - \delta$ semi open set. Furthermore, if $A$ and $B$ are subsets of a bitopological space $(X, \tau_1, \tau_2)$ and $A \cup B$ is a $\tau_1\tau_2 - \delta$ semi open set, then $A, B$ need not be $\tau_1\tau_2 - \delta$ semi open sets. If $A$ and $B$ are $\tau_1\tau_2 - \delta$ semi closed sets in a bitopological space $(X, \tau_1, \tau_2)$, then $A \cap B$ is also $\tau_1\tau_2 - \delta$ semi closed. But $A \cup B$ may not be a $\tau_1\tau_2 - \delta$ semi closed set. Furthermore, if $A$ and $B$ are subsets of a bitopological space $(X, \tau_1, \tau_2)$ and $A \cap B$ is a $\tau_1\tau_2 - \delta$ semi closed set, then $A, B$ need not be $\tau_1\tau_2 - \delta$ semi closed sets. Similarly, Countable union of $\tau_1\tau_2 - \delta$ semi open set is $\tau_1\tau_2 - \delta$ semi open. And Countable intersection of $\tau_1\tau_2 - \delta$ semi closed set is $\tau_1\tau_2 - \delta$ semi closed.

Any subset $A$ is $\tau_1\tau_2 - \delta$ semi open set in a bitopological space $(X, \tau_1, \tau_2)$ if and only if $A \subseteq \tau_2 - cl(\tau_1 - \delta int(A))$. Similarly, Any subset $F$ is $\tau_1\tau_2 - \delta$ semi closed set in a bitopological space $(X, \tau_1, \tau_2)$ if and only if $A \supseteq \tau_2 - int(\tau_1 - \delta cl(A))$. We can prove the results for $\tau_2\tau_1$ as we did for $\tau_1\tau_2$. Now, we introduce a result for a product of two sets. Let $A, B$ be the subsets of bitopological spaces $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ respectively. If $A \in \tau_2\tau_1 - \delta$ semi open in $X$ and $B \in \sigma_1\sigma_2 - \delta$ semi open set in $Y$, then, $A \times B \in \tau_2 \times \sigma_1 \;\; \tau_1 \times \sigma_2 - \delta$ semi open set in $(X \times Y, \tau_2 \times \sigma_1 \;\; \tau_1 \times \sigma_2)$. Similarly, we can show that the previous result holds for closed set too.

Now we are going to discuss the properties of $\tau_1\tau_2 - \delta$ semi continuous functions in bitopological spaces. Let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ be two bitopological spaces. Then, a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $\tau_1\tau_2$ –continuous, if the inverse image of each $\sigma_1$–open set in $Y$ is $\tau_1\tau_2$–open set in $X$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $\tau_1\tau_2 - \delta$ continuous, if the inverse image of each $\sigma_1$–open set in $Y$ is $\tau_1\tau_2$– $\delta$ open set in $X$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $\tau_1\tau_2$– $\delta$ semi continuous, if $f^{-1}(V)$ is $\tau_1\tau_2$– $\delta$ semi open set in $X$, for every $\sigma_1 - \delta$ open set $V$ in $Y$. Further, If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ be two $\tau_1\tau_2$– $\delta$ semi continuous functions, then $g \circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)$ need not be a $\tau_1\tau_2$– $\delta$ semi continuous.

Now, we introduce the following result : Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ . Then, $f$ is $\tau_1\tau_2$–$\delta$ semi continuous if and only if $f^{-1}(U)$ is $\tau_1\tau_2$– $\delta$ semi closed in $X$, for each $\sigma_1$ –closed set $U$ in $Y$.
Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be $\tau_1\tau_2$– $\delta$ semi continuous. Then, $\forall$ $\sigma_1$–open set $V$ in $Y$, $\exists$ $\tau_1\tau_2$– $\delta$ semi open set $P$ in $X$ such that $f(P) \subseteq V$.

Furthermore, A constant function $f : (X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ is $\tau_1\tau_2$–$\delta$ semi continuous. Because, Consider a $\sigma_1$ −open set $V$ in $Y$. If $c \in V$, then $f^{-1}(V) = X$ is $\tau_1\tau_2$–$\delta$ semi open. If $c \notin V$, then $f^{-1}(V) = \phi$ is $\tau_1\tau_2$– $\delta$ semi open. Since $X,\phi$ are $\tau_1\tau_2$– $\delta$ semi open sets in $X$ , $f$ is $\tau_1\tau_2$– $\delta$ semi continuous.

Finally, we are going to define the homeomorphism in bitopological spaces. Let $f : (X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$. Then, $f$ is homeomorphism iff the maps $f_1 : (X,\tau_1) \rightarrow (Y,\sigma_1)$ and $f_2 : (X,\tau_2) \rightarrow (Y,\sigma_2)$ are homeomorphism. To prove this result, suppose that $f$ is homeomorphism. Then, $f$ is continuous and bijective. Further, $f^{-1}: (Y,\sigma_1,\sigma_2) \rightarrow (X,\tau_1,\tau_2)$ exists and continuous. Since $f$ is continuous, $f_1$ and $f_2$ both are continuous. Clearly, $f_1$ and $f_2$ are bijective. Since, $f^{-1}$ is continuous, both $f_1^{-1}$ and $f_2^{-1}$ are continuous. Similarly, we can prove that the converse part of this result is also true.

Let $f : (X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ be bijective and homeomorphism. Then, $f$ is continuous and closed.
Since $f$ is homeomorphism, $f$ and $f^{-1}$ both are continuous. Let $U$ be an open set in $(X,\tau_1,\tau_2)$. Then, $(f^{-1})^{-1}(U) = f(U)$ is open set in space $(Y,\sigma_1,\sigma_2)$. Thus, $f$ is open map. Let $F$- a closed set in space $(X,\tau_1,\tau_2)$. This implies, $X \backslash F$ is open in $(X,\tau_1,\tau_2)$. So, $f(X \backslash F)$ is open in $(Y,\sigma_1,\sigma_2)$. But $f(X \setminus F) = f(X) \setminus f(F) = Y \setminus f(F)$. This implies, $Y \setminus f(F)$ is open in $(Y,\sigma_1,\sigma_2)$. Therefore, $f(F)$ is closed in space $(Y,\sigma_1,\sigma_2)$. i.e. $f$ is closed.

**Conclusions:** In this paper, Some results of $\tau_1\tau_2 - \delta$ semi open sets / closed sets and $\tau_1\tau_2 - \delta$ semi continuous functions in bitopological spaces have been discussed. Furthermore, we have introduced the homeomorphism of bitopological spaces. In addition, we have investigated the relationship between open sets (closed sets) and homeomorphism in bitopological spaces. We plan to extend our research work to uniform continuous, $\tau_1\tau_2$ −connectedness and $\tau_1\tau_2 - \delta$ compactness. Further, we are interested to find some interesting results in bitopological spaces.

**Acknowledgment:** I wish to express my sincere thanks to my supervisor, K.Kannan, Senior lecturer in Mathematics, University of Jaffna, SriLanka.

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