MIXED INITAL BOUNDARY VALUE PROBLEM FOR LAX EQUATION

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Abstract: We consider mixed initial boundary value problem for the following partial differential equation

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\log(b+ae^{u})=0$$

We single out these type of equations as Lax type equations because Lax first considered these type of equations where finite difference scheme applied to such an equation gets linearized and establishing convergence gets simplified. Hypothesis for writing explicit formula for solution of mixed initial boundary value problem for Lax equation using Hamilton Jacoby theory is not satisfied. Despite of this we prove that Finite Difference Scheme converges to the explicit formula given by the Hamilton Jacoby theory.

Introduction: We consider partial differential equation as follows

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\log(b+ae^{u})=0 \qquad (1.1)$$

$$u(x,0)=u_0(x)$$

$$u(0,t)=\lambda(t)$$

Where $a,b>0$. Because of this condition solutions doesn't admit boundary layer. Explicit formula in the case of f (u) convex function of its argument and satisfying additional condition that $\frac{f(u)}{(u)}\to\infty$ is discussed in famous paper by Conway and Hopf [3]. It is stressed in that paper that with not all the forms of λ(t) that explicit solution is possible and boundary condition must satisfy certain conditions. For example λ(t) should satisfy that $\lambda'(t)$ must take values in the range of f(u) in order that explicit formula should exist for mixed initial boundary value problem with flux function f(u). This condition is not satisfied by equation (1.1) and hence in this case it becomes essential that we write explicit formula and prove that this explicit formula is indeed a solution to this mixed initial boundary condition. The boundary condition is prescribed in the sense of Bardos, Leroux and Nedelec[7] so that existence of solution is assured.

Explicit Formula for the Solution of Lax Equation: For each (x, y, t), x ≥ 0; y ≥ 0; t ≥ 0, C(x, y, t) denotes the class of paths β in z - s plane
D = {(z, s) : z ≥ 0, s ≥ 0}
Each path connecting the point (y, 0) to (x, t) and is of the form
z = β(s)
where β is a piecewise linear function with one straight line or many straight lines having slopes of value lying between zero and one. Without loss of generality we can assume that this piecewise linear function contains either single piece of straight line joining (y,0) to (x, t) or pair of straight lines joining first (0, 0) joining to (0, s) and then (0, s) joining to (x, t). We assume that $u_0(x)\in L^1(0,\infty)$and λ(t) be a boundary condition which we assume to be a constant λ to begin with.

Theorem 1: For each fixed x ≥ 0, t ≥ 0 define

$$U(x,t)=\max_{\substack{\beta\in C(x,y,t)\\ y\geq o}}\int_y^\infty u_\upsilon(z)dz+\int_{\{s;\,\beta(s)=0\}}f(\lambda(s)ds - \int_{\{s;\,\beta(s)\neq 0\}} f^*\left(\frac{d\beta}{ds}\right)ds\} \qquad (2.2)$$

U(x; t) is a solution of mixed initial boundary value problem for scalar conservation law (1.1)
Complete article is devoted to prove this theorem. We first consider constant boundary data. For constant boundary data, by following Joseph and Gowda [6] we introduce finite difference scheme and prove that the solution of this finite difference scheme converges to U(x; t) stated in the theorem. Subsequently we prove theorem for non constant boundary data by again following Joseph and Gowda [6] . we subject our scalar conservation law to the following finite difference scheme.

$u(x,t+\Delta) = u(x,t) - \{f\big(u(x,t)\big) - f\big(u(x-\Delta)\big)$ (2.3)
with the initial value $u(x,0) = u_0(x)$ and boundary condition u(o, t) = λ(t) and where Δ is increment in x as well as in t. It is in the nature of the conservation law we are studying that transformation can be found which linearizes the scheme which we will do now.

$$u_k^{n+1} = u_k^n - \ (f(u_k^n) - \ f(u_{k-1}^n)) \qquad (2.3)$$

$$\sum_{y=k}^{\infty} u_k^{n+1} = \sum_{j=k}^{\infty}(u_k^n - (f\big(u_j^n\big) - f\big(u_{j-1}^n\big)))$$

$$U_k^n = \sum_{y=k}^{\infty} u_j^n$$

$$U_k^{n+1} = \ u_k^n + \ f\ (u_{k-1}^n)$$

$$U_k^{n+1} = \ u_k^n + \ f\ (u_{k-1}^n - U_k^n)$$

Let U_k^n = log V_k^n then it follows that $U_{k-1}^n - U_k^n = \log\frac{u_{k-1}^n}{V_k^n}$

$\log V_k^{n+1} = \log V_k^n + \log\ (b + ae^{\log\frac{V_{k-1}^n}{V_k^n}})$ (2.4)

which implies that
$V_k^{n+1} = bV_k^n + aV_{k-1}^n$ (2.5)

which is linear in V_k^n and V_{k-1}^n Thus the transformation from U_k^n to V_k^n linearizes the difference scheme.
We have $\lambda = u_0^n$ the following formula
when n ≤ k

$$V_k^n = e^{\lambda}V_1^n$$

We have for V_k^n the following formula
when n ≤ k
$V_k^n = \ \sum_{i=0}^{n}\binom{n}{i}b^1 a^{n-1}\, V_{k-n+1}^0$ (2.6)
and when n > k

$V_k^n = \ \sum_{i=0}^{k-1}\binom{n}{i}b^{n-i}a^1\, V_{k-n+1}^0 + \sum_{i=0}^{n-k}\binom{n-i-1}{k-1}b^{n-k-i}a^k V_0^i$ (2.7)

We now calculate the expression to which V_k^n converges when $\Delta \rightarrow$ o. We begin with case n < k.

Case 1:$n \leq k$

$$V_k^n = \sum_{i=0}^{n}\binom{n}{i}b^1 a^{n-1}\, V_{k-n+1}^0$$

Let $k - n + l = j$. Then $l = 0 \Longrightarrow j = k - n$ and $l = n \Longrightarrow j = k$ Above equation then becomes

$V_k^n = \ \sum_{j=k-n}^{k-1}\binom{n}{n-k-j} a^{k-j} b^{n-k+j}\, V_j^0$

Denote j[th] summand in this expression by Φ_j. Sterling asymptotic formula for n! is given by

$$n!\ \approx \sqrt{2\pi}\, e^{-n} n^{n+\frac{1}{2}}$$

By subjecting to Sterlings asymptotic formula gives

$\binom{n}{n-k-j} \approx \frac{n^{n+\frac{1}{2}}}{(k-j)^{(k-j)+1/2}\ (n-kj)^{(n-k+j)+1/2}\sqrt{2\pi}}$ (2.8)
Multiply numerator and denominator by $\Delta\frac{n+1}{2}$ and rearranging these terms we get

$$\Phi_j \approx \frac{(\Delta n)^n b^{\Delta(n-k+j)\Delta} a^{\Delta(k-j)/\Delta}}{((k-j)\Delta)^{k-j}\,((n-k+j)\Delta)^{n-k+j}}$$

$$\text{x}\sqrt{\frac{n}{(2\pi(k-j)(n-k+j))}} \qquad (2.9)$$

Suppose this asymptotic expression for Φ_j takes its maximum at n = N; k = K and j = J provided nΔ, lΔ and kΔ remain fixed in the limit as $\Delta \to 0$. Note that Φ_j can be written as follows.

$$\Phi_j \approx \frac{e^{\frac{\Delta n\, log(\Delta_e^{1/2\ log\, n})}{\Delta}}\ e^{\frac{\Delta(n-k+j)\log b}{\Delta}}\ \ e^{1/2\ log\, n}}{e^{\frac{\Delta(k-j)\log(\Delta(k-j)}{\Delta}}\ e^{\frac{\Delta(n-k+j)\ \log(\Delta(n-k+j)}{\Delta}}}$$

$$x\frac{e^{\frac{\Delta(k-1)\log n}{\Delta}}}{e^{1/2\ \log(2\pi(k-j)(n-k+j))}} \qquad (2.10)$$

Note that the term $e^{\frac{\Delta n \log(\Delta n)}{\Delta}}$ achieves at j = J; k = K and n = N. The term $e^{\frac{\Delta(n-N)\log\ (\Delta(n-N)}{\Delta}}$ never becomes positive except at j = J; k = K and n = N it becomes 1. Therefore as Δ is made to approach zero, power of e becomes negative invite and approaches $e^{\frac{\Delta(n-N)\log\ (\Delta(n-N)}{\Delta}}$ zero and does not contribute to the sum. Same thing happens with other terms and in the limit as$\Delta \to 0$. V_k^n then becomes the following.

$$V_k^n = \max_j \Phi_j \qquad (2.11)$$

Δ log V_k^n therefore becomes

$$\Delta \log V_k^n = \max_{k-n<j<k} \Delta n \log(\Delta n) - (\Delta(k-l)\log(\Delta(k-l)$$

$$-\Delta(n-k+l)\log\big(\Delta(n-k+l)\big)$$

$$+(\Delta(n-k+l)\log b + \Delta(k-l)\log a$$

$$+\Delta\log\sqrt{\frac{n}{(2\pi(k-l)(n-k+l)}} + \Delta\log V_1^0$$

We subject this expression to the limit $\Delta \to 0$. We have in this limit Δn = t, Δ k = x and Δt y. With this the above expression, which we denote by A(x, t, y), becomes

A(x, t, y) = t log t - (x - y) log(x -y) - (t - x + y) log(t - x + y)

$$+(t - x + y)\log b + (x - y)\log a + \int_y^{\infty} u_0(x)dx \qquad (2.12)$$

where we have used that in the limit $\Delta \to 0$ the following term becomes zero.

$$+\Delta\log\sqrt{\frac{n}{(2\pi(k-l)(n-k+l)}}$$

Case 2: n > k

$$V_k^n = \Psi_k^n + \Omega_k^n \qquad (2.13)$$

Where

$$\Psi_k^n = \sum_{i=0}^{k-1}\binom{n}{i} b^{n-i}a^i V_{k-i}^0$$

$$\Omega_k^n \sum_{i=0}^{k-1}\binom{n-i-l}{k-l} b^{n-k-i}a^k V_0^i$$

Individual term in Ψ_k^n is denoted by Γ_i. Transformation i + 1 = k - j and then j + 1 = l brings Γ_i to Γ_i where

$$\Gamma_i = \binom{n}{k-i} b^{n-k+1} a^{k-1} V_1^0$$

Above j varies from 0 to k - and l varies from 1 to k. Applying Sterling asymptotic formula and rearranging the terms we get

$$\Gamma_i = \frac{(\Delta n)^{(n+\frac{1}{2})} \Delta^{\frac{1}{2}} b^{\Delta(n-k+l)/\Delta a \Delta(k-l)/\Delta}}{\sqrt{2\pi}((k-l)\Delta)^{(k-l+\frac{1}{2})}(n-k+l)\Delta)^{(n-k+l+\frac{1}{2})}} V_1^0 \quad (2.14)$$

Suppose this asymptotic expression for Γ_i. takes its maximum at n = N, k = K and l = L provided nΔ, kΔ and l Δ remain fixed in the limit $\Delta \to 0$. After some rearrangement note that log Γ_i. can be written as follows.

$$\log \Gamma_i = \frac{e^{\frac{\Delta n \log(\Delta e^{1/2 \log n})}{\Delta}} e^{\frac{\Delta(n-k+l)\log b}{\Delta}} e^{\frac{\Delta(k-l)\log a}{\Delta}} e^{1/2 \log n}}{e^{\frac{\Delta(n-k+l)\log(n-k+l)}{\Delta}} e^{\frac{\Delta(k-l)\log(k-l)}{\Delta}} e^{\frac{1}{2}\log(2\pi(k-l)(n-k+l))}} \quad (2.15)$$

Note that the term $e^{\frac{\Delta n \log(\Delta n)}{\Delta}}$ achieves its maximum at n = N, k = K and l = L and the term $e^{\frac{\Delta(n-N \log \Delta(n-N)}{\Delta}}$ never becomes positive except at n = N, k = K and l = L where it becomes 1. Therefore as Δ is made to approach zero, power of e becomes negative infinite and the term itself becomes zero. Thus as $\Delta \to \mathbf{0}$ we get the following.

$$\Psi_k^n = \max_i \Gamma_i \quad (2.16)$$

Thus we get

$$\Delta \log \Psi_k^n = \max_{l<l<k} [\Delta n \log(\Delta n) - \Delta(k-l)\log(\Delta(k-l)]$$

$$-\Delta(n-k+l)\log(\Delta(n-k+l))$$

$$+\Delta(n-k+l)\log b + \Delta(k-l)\log a \quad (2.17)$$

$$+\Delta\log\sqrt{\frac{n}{2\pi(k-l)(n-k+l)}} + \Delta \log V_1^0]$$

We will now subject this expression to the limit $\Delta \to 0$ In this limit Δn = t, Δk = x and Δl = y and we again denote this expression by A(x, t, y)

A(x, t, y) = t log t - (x - y) log(x - y) - (t - x + y) log (t - x + y)

$$+(t - x + y)\log b + (x - y)\log a + \int_y^\infty u_0(x)dx \quad (2.18)$$

where we have used that in the limit $\Delta \to 0$ the following term becomes zero.

$$+\Delta\log\sqrt{\frac{n}{2\pi(k-l)(n-k+l)}}$$

Now we consider Ω_k^n

$$\Omega_k^n = \sum_{i=0}^{n-i-l} \binom{n-i-l}{k-l} b^{n-k-i} a^k V_0^i \quad (2.19)$$

by denoting each term in this expression by Λ_i , we get on the same lines as we did with Ψ_k^n

$$\Omega_k^n = \max_i \Lambda_i \quad (2.20)$$

In this case we have the following

$$\Delta \log \Omega_k^n = \max_{l<i<n<k} [\Delta(n-i-l)\log(\Delta(n-i-l)]$$

$$-\Delta(k-l)\log(k-l)) \quad (2.21)$$

$$-\Delta(n-i-l)\log(\Delta(n-i-k))$$

$$+\Delta(n-i-k)\log b + \Delta k \log a + \Delta \log V_0^i]$$

Let's introduce the following Δn = t, Δ(k-l) = x and Δ (i + 1) = s. With these specifications the above expression, which we denote by B(x, t, s), becomes

B(x, t, s) = (t - s) log (t - s) - x log x - (t - x - s) log (t - x - s)
$+\left(t-s-x\right)\log b \;+\; s\log \mathrm{a} \;+\int_y^s f(\lambda)ds$ (2.22)
Where we have made use of the following relation
$V_0^i = f(\lambda)^i V_0^0$ (2.23)
and $\Delta \log V_0^0 \to 0$ as $\Delta \to 0$. Let A(x, t, y) be denoted as $\phi(y)$ to emphasize dependence on y. $\phi(y)$ takes maximum value at $y = x\frac{a}{a+b}t$. With this value of y, ϕ becomes t log(a + b) and we prove that as $\Delta \to 0$, $\log V_k^n \to U\ (x,t)$ where U(x; t) is as stated in the theorem. Let $\lambda(t)$ be a step functions follows.

$\lambda(t) = \lambda_j$ for $t_{j-1} < t < t_j$, j = 1, 2..k. $0 < t_1 < t_2 < \dots < t_k < T$

Here λ_j are constants. We take initial time as t_j, and get for $t_j < t < t_{j+1}$ the following

$$\lim_{\Delta\to 0} \log V_k^n = \max_{\substack{B\in C(x,y,t)\\ y\geq 0}} \{U_j(y_j) + \int_{\{s:\beta(s)=0\}} f\overline{(\lambda(s)}ds - \int_{\{s:\beta(s)\neq 0\}} f^*\left(\frac{d\beta}{ds}\right)ds\} \quad (2.23)$$

where Uj(yj) = U(yj ; tj) Here C(x; yj ; t; tj) denote the class of the paths which connect (yj ; tj) to point (x; t) with slope of value less than or equal to one. Thus we get

$$\lim_{\Delta\to 0} \log V_k^n = \max_{\substack{B\in C(x,y,t)\\ y\geq 0}} \{U_0(y) + \int_{\{s:\beta(s)=0\}} f\overline{(\lambda(s)}ds - \int_{\{s:\beta(s)\neq 0\}} f^*\left(\frac{d\beta}{ds}\right)ds\} \quad (2.24)$$

Now we claim that maximum is achieved for some $\beta \in$ C(x, y, t). But this follows from the fact that as long as β (s) does not touch t - axis maximum is achieved for path having slope less than or equal to one. Further from the expression for U(x, t) value of maximum is increased by diminishing value of y1, therefore such a path obtained by diminishing value of y1 can not maximize the required expression, and it follows that maximizing path touch the s - axis atmost once. On the same lines as these we can prove the following theorem
Theorem 2. Let $\underline{V_k^n}$ and $\overline{V_k^n}$ be the solutions of finite difference scheme with λ(t) replaced by $\underline{\lambda(t)}$ and $\overline{\lambda(t)}$ respectively, where $\underline{\lambda(t)} \leq \overline{\lambda(t)}$ then $\underline{V_k^n} \leq \overline{V_k^n}$

Theorem 3: Let V_k^n be the finite difference solution of mixed initial boundary value problem with $\lambda(t)$ as a continuous function of t. Then it converges to true solution of mixed initial boundary value problem.
Proof: Since $\lambda(t)$ is continuous we can construct step functions $a_n(t)$ and $b_n(t)$ such that
$a_n(t) \leq \lambda(t) \leq b_n(t)$
and $a_n(t)$ and $b_n(t)$ converge uniformly to $\lambda(t)$ in [0, T]. Let $A_n^\Delta(x,t)$ and $B_n^\Delta(x,t)$
so that
$\Delta \log A_n^\Delta(x,t) \leq \Delta \log V_k^m \leq \Delta \log B_n^\Delta(x,t)$ (2.25)
Using now results on step functions for step functions an(t) and bn(t) we get

$$\max_{\beta\in C(x,y,t)} \{\int_y^\infty u_0(z)dz + \int_{\{s:\beta(s)=0\}} f(b_{n)}(s)ds - \int_{\{s:\beta(s)\neq 0\}} f^*\left(\frac{d\beta}{ds}\right)ds\}$$

$$\leq \lim_{\Delta\to 0}\ inf\,U_n^\Delta(x,t)$$
$$\leq \lim_{\Delta\to 0}\ sup\ U_n^\Delta(x,t)$$

$$\leq \max_{\beta\in C(x,y,t)} \{\int_y^\infty u_0(z)dz + \int_{\{s:\beta(s)=0\}} f(a_{n)}(s)ds - \int_{\{s:\beta(s)\neq 0\}} f^*\left(\frac{d\beta}{ds}\right)ds$$

Letting $n \to \infty$ and then letting $\Delta \to 0$ we get

$$U(x,t) = \max_{\substack{\beta\in C(x,y,t)\\ y\geq 0}} \{\int_y^\infty u_0(z)dz + \int_{\{s:\beta(s)=0\}} f\big(\lambda(s)\big)\,ds$$

$$-\int_{\{s:\beta(s)\neq 0\}} f^*\left(\frac{d\beta}{ds}\right)ds \quad (2.26)$$

and hence the proof.

Conclusions: Writing explicit formula for mixed initial boundary value problems, in general, is not possible. Using Hamilton-Jacobi theory it is possible only in case of Lax type equations. We have proved in this article that although hypothesis required to apply Hamilton Jacobi theory is not fulfilled still explicit formula can be written and we have proved that such a solution is achieved in the limit of finite difference scheme. We could prove this result only because finite difference scheme can be linearized.

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