MIXED INITIAL BOUNDARY VALUE PROBLEM FOR LAX EQUATION: AN ALTERNATIVE APPROACH

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Abstract: In continuation with our investigations on constructions of explicit formulae for scalar convex conservations laws we consider mixed initial boundary value problem for Lax equation. We have already derived explicit formula for this problem. It appears that calculations are complicated and are not illuminating enough to understand recursive nature of derivation. In this paper we provide an alternative derivation for the explicit formula for Mixed Initial Boundary Value Problem for Lax equation.

Introduction:

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\log(b+ae^{u})=0 \qquad (1.1)$$
$$u(x,0)=u_0(x)$$
$$u(0,t)=\lambda(t)$$

Where we take $a,b>0$ so that solution doesn't admit boundary layer. In [1] on the lines of derivations provided in [6] and [7] we derived explicit formula for the solution of mixed initial boundary value problem (1.1). We proved that solution of a finite difference scheme converges to the solution of (1.1) in the limit $\Delta\to 0$. Where Δ is a step length used in the derivations of finite difference scheme for the problem (1.1). Details about finite difference scheme are given in [1], [6] and [7]. We use the same notations as those are given in [1]. We produce, on the lines of [6],[7] alternative derivation of V_k^n. The boundary condition is prescribed in the sense of Bardos, Leroux and Nedelec[8] so that existence of solution is assured.

Derivation of Alternative Formula for V_k^n: Note that in the case $n\le k$ formula is already in simple form and it is discussed in [1]. In the following article we consider the case $n>k$ and on the lines of [6] and [7] we attempt to simplify formula for V_k^n

$$V_k^n=\sum_{i=0}^{k-1}\binom{n}{i}b^{n-i}a^iV_{k-i}^0+\sum_{i=0}^{k-1}\binom{n}{k-1}b^{n-k-i}a^kV_0^i \qquad (2.1)$$

$$V_{k+1}^n=\sum_{i=0}^{k-1}\binom{n}{i}b^{n-i}a^iV_{k+1-i}^0+\sum_{i=0}^{k-1}\binom{n}{k-1}b^{n-k-i}a^kV_1^i$$

From these two equations we get the following-

$$V_k^n=e^{\lambda}V_{k+1}^n+\sum_{i=0}^{k-1}\binom{n}{i}b^{n-i}a^i(V_{k-i}^0-e^{\lambda}V_{k+1-i}^0) \qquad (2.2)$$

We rewrite the same equation

$$V_k^n=e^{\lambda}V_{k+1}^n+S_{n,k}$$

Where

$$S_{n,k}=\sum_{i=0}^{k-1}\binom{n}{i}b^{n-i}a^i(V_{k-i}^0-e^{\lambda}V_{k+1-i}^0)$$

After a few manipulations we get the following

$$V_k^n=e^{(n-k)\lambda}V_n^n+\sum_{j=0}^{n-k-1}e^{\lambda j}S_{n,k+j} \qquad (2.3)$$

Which after plugging expression for $S_{n,k+j}$ and simplifying becomes the following equation

$$V_k^n=e^{(n-k)\lambda}+\sum_{l=0}^{n}\binom{n}{l}b^{n-l}a^lV_l^0+\sum_{j=0}^{n-k-1}e^{\lambda j}\sum_{i=0}^{k+j-1}\binom{n}{i}b^{n-i}a^iV_{k+j-i}^0-\sum_{j=0}^{n-k-1}e^{\lambda(j+1)}\sum_{i=0}^{k+j-1}\binom{n}{i}b^{n-i}a^iV_{k+j+1-i}^0$$

After performing few more simple manipulations we get the following:

$$V_k^n=e^{(n-k)\lambda}+\sum_{l=0}^{n}\binom{n}{l}b^{n-l}a^lV_l^0+e^{\lambda}\binom{n}{k}b^{n-k}a^kV_1^0+e^{2\lambda}\binom{n}{k+1}b^{n-k-1}a^{k+1}V_1^0+e^{3\lambda}\binom{n}{k+1}b^{n-k-2}a^{k+2}V_1^0$$
$$+\sum_{j=4}^{n-k-1}e^{\lambda j}\sum_{i=0}^{k+j-1}\binom{n}{l}b^{n-i}a^iV_{k+j-i}^0+\sum_{j=3}^{n-k-1}e^{\lambda(j+1)}\sum_{i=0}^{k+j-1}\binom{n}{l}b^{n-i}a^iV_{k+j+1-i}^0$$

Which upon further simplification becomes the following:

$$V_k^n=e^{(n-k)\lambda}+\sum_{l=0}^{n}\binom{n}{l}b^{n-l}a^lV_l^0+\sum_{i=0}^{n}\binom{n}{i}b^{n-i}a^iV_{k-i}^0+\sum_{j=0}^{n-k-2}e^{(j-1)\lambda}\binom{n}{k+j}b^{n-k-j}a^{k+j}V_1^0-e^{(n-k)\lambda}\sum_{i=0}^{n-2}\binom{n}{i}b^{n-i}a^iV_{n-i}^0 \qquad (2.4)$$

Basic equation in the above formula is of the following form-

$$\log\binom{n}{l}a^{n-l}b^l$$

Using Sterling's asymptotic formula for log(n!) we get on the same lines of [1] the following formula $\log\binom{n}{l}a^{n-l}b^l \approx -n(\log\frac{n-l}{n} + \frac{l}{n}(\log\frac{l}{n-l} - \log a + \frac{l}{n}\log ba)) + \frac{1}{2}\log 2\pi + \frac{1}{2}\log\frac{n}{l(n-l)}$ (2.5)

Multiply above equation by Δ to get

$$\Delta\log\binom{n}{l}a^{n-l}b^l \approx -n\Delta(\log\frac{(n-l)\Delta}{n\Delta} + \frac{l\Delta}{n\Delta}(\log\frac{l\Delta}{(n-l)\Delta} - \log a + \frac{l\Delta}{n\Delta}\log ba)) + \frac{1}{2}\Delta(\log 2\pi + \frac{1}{2}\log\frac{n\Delta}{l(n-l)\Delta})$$

As $\Delta \to 0$, last term in the above expression, which is error left in using Stirling's formula approaches zero and only part which remains is

$$-n\Delta(\log\frac{(n-l)\Delta}{n\Delta} + \frac{l\Delta}{n\Delta}\log\frac{l\Delta}{(n-l)\Delta}) \qquad (2.6)$$

Let $\Delta n = t$ and $\Delta l = y$ then we get four separate expressions as

- $\max_{0\le y\le t} -t\log\left(1-\frac{y}{t}\right) - \frac{y}{t}\log\left(\frac{t-y}{y}\frac{b}{a}\right) + \int_y^\infty u_0(z)dz + (t-x)\lambda$
- $\max_{0\le y\le t} -t\log\left(1-\frac{y}{t}\right) - \frac{y}{t}\log\left(\frac{t-y}{y}\frac{b}{a}\right) + \int_y^\infty u_0(z)dz$
- $\max_{0\le s\le t} s\lambda - t\log\left(1-\frac{x+s}{t}\right) + \frac{x+s}{t}\log\left(\frac{t-x-s}{x+s}\frac{b}{a}\right) + \int_y^\infty u_0(z)dz$
- $\max_{0\le y\le t} -t\log\left(1-\frac{y}{t}\right) - \frac{y}{t}\log\left(\frac{t-y}{y}\frac{b}{a}\right) + \int_{t-y}^\infty u_0(z)dz + (t-x)\lambda$

And the solution to the mixed initial boundary value problem for Lax equation is maximum of these four expressions.

Conclusions: Finding explicit formula for mixed initial boundary value problems for scalar conservation laws is very difficult and it depends on functional form of flux function. Lax equation is one of the simplest kind as in this case solution doesn't admit boundary layer. On the lines of treatment given to such flux functions by Joseph and Gowda in [6] and [7] we have proved that solution of finite difference scheme converges to mixed initial boundary value problem for Lax equation.

Appendix: In this section we will discuss a few specific initial boundary value problem. We will give solutions to the problem when initial data is of Riemann type. We will keep boundary data constant.

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\log(b + ae^u) = 0$$
$$u(x,0) = u_0(x)$$
$$u(0,t) = \lambda(t)$$
$$u(x,0) = \begin{cases} 1, & 0 < x \le 10 \\ 0, & 10 < x \end{cases}$$

Pure initial value problem (i.e without boundary condition) admits simple wave solution. Slope of a characteristic is

$$\frac{dx}{dt} = \frac{ae^u}{b + ae^u}$$

(1) Let us impose boundary condition $\lambda = 1$. Then solution of the problem is

$$u(x,t) = \begin{cases} 1 & x - 10 < \sigma t \\ 0 & \sigma t \le x \end{cases}$$

Where

$$\sigma = \frac{(f(u_l) - f(u_r))}{(u_l - u_r)} = 0.1282882$$

(2) $\lambda = 0.5$

In this case solution is

$$u(x,t) = \begin{cases} 0.5 & x < \tau t \\ g'\left(u\left(\frac{x}{t}\right)\right) & \tau t < x < \sigma t \\ 1 & \sigma t < x - 10 < \psi t \\ 0 & \psi t < x - 10 \end{cases}$$

Where $\tau = 3.426122$, $\sigma = 0.82321499$, $\psi = 0.1282882$

Both these illustrations are just illustrative and such solutions can always be constructed in variety of simple situations.

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