# ON $|\bar{N}, p_k|$-SUMMABLE SEQUENCE SPACE DEFINED BY ORLICZ & MODULUS FUNCTIONS AND ITS CONVERGENCE

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**Abstract:** The motivation of this work is to introduce and study the generalized sequence space $|\bar{N}_p|_\theta(X_{\varphi(a)}, M, F, r, s)$ defined by using the concepts of lacunary sequence $\theta = \{k_r\}$, $|\bar{N}, p_k|$- -summability, the sequence $M = (M_i)$ of Orlicz functions, the sequence $F = (f_i)$ of modulus functions and multiplier sequence $(i^{-s})$ , $s \geq 0$. Further, a paranorm structure has been imposed and the concept of $|\bar{S}|_\theta$-lacunary statistically convergence with respect to $|\bar{\mathbf{N}}, \mathbf{p_k}|$- summability has been studied on this sequence space.

**Keywords:** Lacunary Convergence, Lacunary Statistical Convergence, $|\bar{\mathbf{N}}, \mathbf{p_k}|$-Summability Paranormed Space.

**Introduction:** Let $\sum_{k=0}^{\infty} a_k$ be an infinite series with the sequence of partial sums $(s_k)$. Let $(p_k)$ be a sequence of positive real numbers and $P_k = \sum_{k=0}^{n} p_n$. The series $\sum_{k=0}^{\infty} a_k$ is said to be $|\bar{\mathbf{N}}, \mathbf{p_k}|$ -summable [1] to the finite limit $\ell$ if

$t_k = \frac{1}{P_k}\sum_{\nu=0}^{k} p_\nu\, s_\nu \to \ell$ as $k \to \infty$

and is said to be absolutely $|\bar{\mathbf{N}}, \mathbf{p_k}|$ -summable if $\sum_k |t_k - t_{k-1}| < \infty$.

In [7] it is shown that, given a sequence $a = (a_n)$ and for $k \geq 1$,

$$\phi_k(a) = t_k - t_{k-1} = \frac{p_k}{P_k P_{k-1}} \sum_{n=1}^{k} P_{n-1}\, a_n,$$

Note that, for any sequences $a, b$ and scalar $\lambda$, we have $\phi_k(a + b) = \phi_k(a) + \phi_k(b)$ and $\phi_k(\lambda a) = \lambda\phi_k(a)$.

An Orlicz function [17] is a function $M: (0, \infty] \to (0, \infty]$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$. If the convexity condition is replaced by the condition $M(x + y) = M(x) + M(y)$, then the function is called modulus function [28].

An Orlicz function is said to satisfy $\Delta_2$ −condition [19] for all values of $x$, if and only if $M(2x) \leq KM(x)$ for $K > 0$, $x \geq 0$.

This condition is equivalent to $M(Lx) \leq KLM(x)$, $\forall x \geq 0, \forall L \geq 1$. Also an Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for $0 < \lambda < 1$ which is again equivalent to the inequalitity $\alpha x \leq M(x) \leq \beta(x)$ for $\alpha, \beta > 0, x \geq 0$.

Introduction of modulus function [28] and Orlicz function [17] has given a new dimension in the development of the theory of sequence spaces. Ruckle [28] used the idea of modulus function $f$ to define a new sequence space $\ell$ (f) where

$\ell(\mathrm{f}) = \{\mathrm{x} = (\mathrm{x_k}): \sum_{k=1}^{\infty} \mathrm{f}(|\mathrm{x_k}|) < \infty\}$.

whereas Tzafriri and Lindenstrauss [20] used the idea of Orlicz function $M$ to construct the Orlicz sequence space

$l_M = \left\{\mathrm{x} = (\mathrm{x_k}): \sum_{k=1}^{\infty} \mathrm{M}(\frac{|\mathrm{x_k}|}{\rho}) < \infty,\ \text{for some } \rho > 0\right\}$.

which becomes a Banach space with the norm $\|x\| = inf\left\{ \rho > 0: \sum_{k=1}^{\infty} \mathrm{M}\left(\frac{|\mathrm{x_k}|}{\rho}\right) \leq 1 \right\}$

Later on, Various sequence spaces defined by Orlicz function and modulus function have been developed and discussed by Mursaleen et al. [23], Choudhury *et al.* [10], Güngör *et al.* [16], Nurray *et al.* [24], Ghosh *et al.* [15], Basu et al. ([3], [4]), Savas [32] and many others..

Using the idea of Orlicz function and the concept of $|\bar{N}, p_k|$ summability, Bhardwaj and Singh [7] introduced and studied the following class

$|\bar{N}_p|(M,r,) = \left\{ a = (a_k): \sum_{k=1}^{\infty} \left[ M(\frac{|\phi_k(a)|}{\rho}) \right]^{r_k} < \infty,\ \text{for some } \rho > 0 \right\}$, where
$r = (r_k)$ be a bounded sequence of strictly positive real numbers.
These classes have been further generalized by Altin et. al. [1] by choosing the elements of $a = (a_k)$ in a seminormed space $X$ with seminorm $q$,
$|\bar{N}_p|(M,r,q,s) = \left\{ a \in w(X): \sum_{k=1}^{\infty} k^{-s} \left[ M\left( q(\frac{|\phi_k(a)|}{\rho}) \right) \right]^{r_k} < \infty,\ \text{for some } \rho > 0 \right\}$
where $r = (r_k)$ is a bounded sequence of strictly positive real numbers.

By a 'lacunary sequence' [12] we mean an increasing sequence of positive integers $\theta = (k_r)$ where $k_0 = 0$, and $0 < k_r < k_{r+1} \to \infty$ as $r \to \infty$ . The intervals determined by $\theta$ are denoted by $I_r = (k_{r-1},\ k_r]$. Freedman et al. [12] introduced the space of lacunary strongly convergent sequences $N_\theta$ as follows:
$N_\theta = \left\{ x = (x_i):\ \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} |x_i - s| = 0,\ \text{ for some } s \right\}$
Statistical convergence for real and complex sequences was first defined by Steinhaus [29] and then H.Fast [11], Buck [8] and Schoenberg [33] independently. Fast extended the concept of sequential limit which he called statistical convergence. Schoenberg gave some basic properties of statistical convergence and studied the concept as summability method.
A sequence $x = (x_k)$ of complex numbers is said to be statistically convergent to $L$ if for any $\epsilon > 0$,
$\lim_{n\to\infty} \frac{1}{n} |\{k \le n: |x_k - L|\} \ge \epsilon| = 0$
where the vertical bar denotes the cardinality of the enclosed set.

From the point of view of sequence spaces this convergence method has been generalized and developed by Fridy [13], Salat [30], Connor [9] and many others. Later on, Fridy and Orhan [14] combined the concepts of statistical convergence and lacunary convergence and introduced a new convergence method known as lacunary statistical convergence. Recently these convergence methods have also been studied on fuzzy sequence spaces by Nurray [25], Savas [31], Basu [6] and many others.

Being motivated by the existing literature, the present author has made an attempt to extend the study on the sequence space $|\bar{N}_p|_\theta(X_{\varphi(a)}, M, F, r, s)$ of sequences of the elements of a Banach space $X$, defined by using sequences of modulus $F = (f_i)$ as well as Orlicz functions $M = (M_i)$. Further lacunary statistical convergence has been studied on this space.

Throughout the work, the following defnition and standard inequalities have been used frequently:

**Paranorm:** Let $X$ be a linear space. Then $g: X \to R$ is called paranorm on $X$ if for $x, y \in X$ and any scalar $\lambda$, (i) $g(x) \ge 0$ (ii) $x = \theta$ implies $g(x) = 0$; (iii) $g(x) = g(-x)$(iv) $g(x + y) \le g(x) + g(y)$; (v) $g(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$, whenever $\lambda_n \to \lambda$ and $x_n \to x$, for scalars $\lambda_n, \lambda$ and vectors $x_n, x$
(for all $n \in \mathbb{N}$) $\in$ X. The space $(X, g)$ is called a paranormed space.
If (i) is replaced by (i) $g(x) = 0$ if and only if $x = \theta$ then $g$ is called a total paranorm on $X$.

**Inequalities:** Let $r = (r_k)$ be a bounded sequence of strictly positive real numbers with $0 < r_k \le \sup r_k = H, D = max\,(1, 2^{H-1}), T = \max(1,\ H)$. Let $f$ be a modulus function.

1. $|a_k + b_k|^{p_k} \le D\{|a_k|^{p_k} + |b_k|^{p_k}\}$ ; [21]
2. $|\lambda|^{p_k} \le \max(1, |\lambda|^H)$; [21]
3. $f(\lambda)^{r_k} \le (1 + [|\lambda|]^H)\, f(1)$, [22]

**The Space $|\bar{N}_p|_\theta(X_{\varphi(a)}, M, F, r, s)$:**
The new sequence space is now introduced as follows:
Let $M = (M_i)$ be a sequence of Örlicz functions satisfying
$\lim_{u\to 0^+} \sup_i M_i\left(\frac{u}{\rho}\right) = 0$ for some $\rho > 0$ … … … … ..**(2.1)**
Let $F = (f_k)$ be a sequence of modulii satisfying
$\lim_{k\to\infty} f_k(u) > 0$, where $u \in [0,\ \infty)$.............**(2.2)**
Let $(X,\ \|.\|)$ be a Banach space over the complex field $\mathbb{C}$ and $r = (r_i)$ be a bounded sequence of strictly positive real numbers with $0 < r_i \le \sup r_i = H,\ D = \max(1, 2^{H-1}), T = \max(1,\ H)$.
Let $X_{f_i(|\phi(a)|)} = \{a = (a_n) \in w(X): f_i(|\phi(a)|) \in X\}$

Where $w(X)$ is the class of sequences of elements of $X$. For simplicity, throughout the article $X_{f_i(|\phi(a)|)}$ will be denoted by $X_{\phi(a)}$.
Let $\theta = (k_j)$ be a lacunary sequence where $k_0 = 0,\ 0 < k_j < k_{j+1}$ and $h_j = k_{j+1} - k_j \longrightarrow \infty$ as $j \to \infty$ .The intervals determined by $\theta$ are denoted by $I_j = (k_{j-1},\ k_j]$.
The following sequence spaces are defined as follows:

$$\left|\overline{\mathrm{N}_{\mathrm{p}}}\right|_{\theta}(X_{\phi(a)}, \mathrm{M}, \mathrm{F}, \mathrm{r}, s) = \left\{a = (a_n): \lim_{j\to\infty} \frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi(a)|-\ell)\|}{\rho}\right)\right]^{r_i} = 0,\ \ s \geq 0 \text{ for some } \rho > 0,\ \ell \in \mathbb{C}\right\}.$$

$$\left|\overline{\mathrm{N}_{\mathrm{p}}}\right|^{0}_{\theta}(X_{\phi(a)}, \mathrm{M}, \mathrm{F}, \mathrm{r}, s) = \left\{a = (a_n): \lim_{j\to\infty} \frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi(a)|)\|}{\rho}\right)\right]^{r_i} = 0,\ \ s \geq 0 \text{ for some } \rho > 0,\ \ell \in \mathbb{C}\right\}.$$

$$\left|\overline{\mathrm{N}_{\mathrm{p}}}\right|^{0}_{\theta}(X_{\phi(a)}, \mathrm{M}, \mathrm{F}, \mathrm{r}) = \left\{a = (a_n): \lim_{j\to\infty} \frac{1}{h_j}\sum_{i\in I_j} \left[M_i\left(\frac{\|f_i(|\phi(a)|)\|}{\rho}\right)\right]^{r_i} = 0,\ \ \text{for some } \rho > 0,\ \ell \in \mathbb{C}\right\}$$

**Particular Case:** Some known sequence spces can be derived from these sequence spaces by restricting $M, F, r$ and $s$ as follows:
Choosing $s = 0,\ M_i = 1,\ \rho = 1,\ f_i = f, \phi_i(a){=}x_i,\ r_i = 1$ for all $i$, $X = \mathbb{R}$ we have the spaces of Pehlivan *et. al.* [27];

$$\mathrm{N}_{\theta}(f) = \left\{x = (x_i): \lim_{r\to\infty} \frac{1}{h_r}\sum_{i\in I_r} f(|x_i - \ell|) = 0 \ \text{ for some } \ell\right\};$$

Choosing $s = 0,\ M_i = I,\ \rho = 1,\ \phi_i(a){=}t_m(\mathrm{a})$, where $t_m(\mathrm{a}) = \frac{a_n + a_{\sigma^1(n)} + \cdots \ldots + a_{\sigma^i(n)}}{i+1}$ , $\sigma^i(n)$ denote the $i$-th iterate of the mapping $\sigma$ at $n$, we have the space of Karakaya *et. al.* [18].;

$$[\mathrm{w}^0_{\sigma}, \mathrm{F}]_{\theta} = \left\{a = (a_n): \lim_{r\to\infty} \frac{1}{h_r}\sum_{i\in I_r} f_i(|t_m(\mathrm{a})|) = 0 \ \text{ uniformly in } \ \mathrm{n}\right\};$$

Choosing $M_i = M,\ f_i = I, X = \mathbb{R},\ \phi_i(a) = x_i,\ r_i = 1$, for all $i$, $\theta = (2^r)$ and s $= 0$ we have the spaces of Parashar *et. al.* [26];

$$W(M, p) = \left\{x = (x_i): \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} M\left(\frac{|x_i - \ell|}{\rho}\right) = 0 \ \text{ for some } \rho > 0\right\};$$

Choosing $M_i = M, f_i = I,$ and $|\gamma^{\mathrm{i}}| = \mathrm{i}^{-\mathrm{s}}, \mathrm{s} \geq 0$ we have the spaces of Basu [5] ;

$$\left|\overline{\mathrm{N}_{\mathrm{p}}}\right|_{\theta}(\mathrm{E}, \mathrm{M}, \mathrm{r}, \gamma) = \left\{x = \{x_k\}: \lim_{j\to\infty} \frac{1}{h_j}\sum_{i\in I_j} |\gamma_i| \left[M_i\left(\frac{\|\phi_i(x)\|_E}{\rho}\right)\right]^{r_i} = 0 \text{ for some } \rho > 0\right\}$$

**Main Results:**

**Theorem 3.1:** $\left|\overline{\mathrm{N}_{\mathrm{p}}}\right|_{\theta}(X_{\phi(a)}, \mathrm{M}, \mathrm{F}, \mathrm{r}, s)$ and $\left|\overline{\mathrm{N}_{\mathrm{p}}}\right|^{0}_{\theta}(X_{\phi(a)}, \mathrm{M}, \mathrm{F}, \mathrm{r}, s)$ are linear spaces over the complex field C, where $M = (M_i)$ and $F = (f_i)$ satisfy conditions 2.1 and 2.2 respectively.

**Proof:** We will prove the result for $\left|\overline{\mathrm{N}_{\mathrm{p}}}\right|^{0}_{\theta}(X_{\phi(a)}, \mathrm{M}, \mathrm{F}, \mathrm{r}, s)$. The other case is similar.
Let $a = (a_n),\ b = (b_n) \in \left|\overline{\mathrm{N}_{\mathrm{p}}}\right|^{0}_{\theta}(X_{\phi(a)}, \mathrm{M}, \mathrm{F}, \mathrm{r}, s)$
and $, \beta \in \mathbb{C}$ .

$$\lim_{j\to\infty} \frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi(a)|)\|}{\rho_1}\right)\right]^{r_i} = 0, \quad \text{for some } \rho_1 > 0,$$

and

$$\lim_{j\to\infty} \frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi(b)|)\|}{\rho_2}\right)\right]^{r_i} = 0, \quad \text{for some } \rho_2 > 0.$$

Let$\rho_3 = \max\ (2\ (1 + [|\alpha|\ ])\rho_1,\ 2\ (1 + [|\beta\ |])\ \rho_2\ )$
Since each $f_i$ is non-decreasing, sub-additive and $M_i$ is non-decreasing and convex,

$$\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi_i(\alpha a + \beta b)|)\|}{\rho_3}\right)\right]^{r_i} \leq$$

$$\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi_i(\alpha a)|)\| + \|f_i(|\phi_i(\beta b)|)\|}{\rho_3}\right)\right]^{r_i}$$

$$\leq \sum_{i\in I_j} \frac{1}{2^{r_i}} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi_i(\alpha a)|)\|}{\rho_1}\right) + M_i\left(\frac{\|f_i(|\phi_i(\beta b)|)\|}{\rho_1}\right)\right]^{r_i}$$

$$\leq \sum_{i\in I_j} i^{-s}\left[M\left(\frac{\|f_i(|\phi_i(\alpha a)|)\|}{\rho_1}\right)+M_i\left(\frac{\|f_i(|\phi_i(\beta b)|)\|}{\rho_2}\right)\right]^{r_i}$$

$$\leq D\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi_i(\alpha a)|)\|}{\rho_1}\right)\right]^{r_i}+D\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi_i(\alpha a)|)\|}{\rho_2}\right)\right]^{r_i}$$

where $D=\max(1,\ 2^{H-1})$. This proves the linearity of the new space.

**Theorem 3.2**: $|\overline{\mathrm{N_p}}|_\theta^0(X_{\phi(a)},\mathrm{M},\mathrm{F},\mathrm{r},s)$ is a total paranormed linear topological space under the paranorm

$$g(a)=\inf\left\{\rho^{r_k/T} : \left[\frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi(a)|)\|}{\rho}\right)\right]^{r_i}\right]^{1/T}\leq 1 \text{ for sufficiently large } j \text{ and } k=1,2,\ldots\right\},$$

$x=(x_k)\in |\overline{\mathrm{N_p}}|_\theta(\mathrm{E},\mathrm{M},\mathrm{r},\gamma)$ where each $M_i$ satisfies $\Delta_2$-condition.

**Proof:** Let $a=(a_n)\in |\overline{\mathrm{N_p}}|_\theta^0(X_{\phi(a)},\mathrm{M},\mathrm{F},\mathrm{r},s)$.
Its clear that $g(a)=g(-a)$ and taking $\alpha=\beta=1$ in the previous theorem we get, $g(a+b)\leq g(a)+g(b)$.
Also, for $a=\theta, g(a)=0$, as $M_i(0)=0$ for each $i$,
where $\theta$ denotes the sequence $(0,0,\ldots.)$. Now suppose that $g(a)=0$.
Then for a given $\varepsilon>0$, there exists some $\rho_\epsilon(0<\rho_\epsilon<\varepsilon)$ such that

$$\left[\frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi_i(a)|)\|}{\varepsilon}\right)\right]^{r_i}\right]^{1/T}$$

$$\leq\left[\frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi_i(a)|)\|}{\rho_\epsilon}\right)\right]^{r_i}\right]\leq 1^{1/T}$$

for each $j$.

If possible, let, $a\neq\theta$ i.e., $a_{n_m}\neq 0$, for some $m\in I_j|$.
Then letting $\varepsilon\to 0$ and using the properties of $\|.\|$, $f_i$ and $M_i's$ we get

$$\left(\frac{\|f_i(|\phi_i(a)|)\|}{\varepsilon}\right)\to\infty$$

and hence

$$\left[\frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi_i(a)|)\|}{\varepsilon}\right)\right]^{r_i}\right]^{1/T}\to\infty$$

which is a contradiction. Hence , $a=\theta$.

Next we will show that, $g$ is continuous with respect to scalar multiplication.
Let a $\in|\overline{\mathrm{N_p}}|_\theta^0(X_{\phi(a)},\mathrm{M},\mathrm{F},\mathrm{r},s)$ and $\lambda\in\mathbb{C}$. To show $g(\lambda a)\to 0$.
Since $g(\lambda a)=\inf\left\{(\rho)^{r_k/T} : \left[\frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi_i(\lambda a)|)\|}{\rho}\right)\right]^{r_i}\right]^{1/T}\leq 1,\right\}$ for sufficiently large $j$ and $k=1,2,\ldots$
We can write,

$$g(\lambda a)=\inf\left\{\begin{array}{c}((1+[|\lambda|])\delta)^{r_k/T} :\\ \left[\frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi_i(a)|)\|}{\delta}\right)\right]^{r_i}\right]^{1/T}\\ \leq 1\end{array}\right\}$$

where $\delta=\frac{\rho}{1+[|\lambda|]}$.
Since $(1+[|\lambda|\,])^{r_k}\leq\max(1,(1+[|\lambda|\,])^T)$,
then $(1+[|\lambda|])^{\frac{r_k}{T}}$
$\leq\{\max(1,(1+[|\lambda|\,])^T)\}^{1/T}$,

Hence $g(\lambda a) = \{\max(1, (1 + [|\lambda|])^T)\}^{\frac{1}{T}} \times$

$$\inf\left\{\begin{array}{c} \vdots \\ (\delta)^{r_k/T} : \\ \left[\frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi_i(a)|)\|}{\delta}\right)\right]^{r_i}\right]^{1/T} \leq 1 \\ \text{for sufficiently large j} \end{array}\right\}$$

Which converges to zero as $a \in |\overline{N_p}|_\theta^0(X_{\phi(a)}, M, F, r, s)$.
Next let $\lambda \to 0$ for some scaler $\lambda \in \mathbb{C}$ and $a = (a_n) \in |\overline{N_p}|_\theta^0(X_{\phi(a)}, M, F, r, s)$.
Then for arbtrary positive number $\epsilon > 0$, there exist a positive integer $\mathbb{N}$ such that

$\frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi_i(a)|)\|}{\rho}\right)\right]^{r_i} < \left(\frac{\epsilon}{2}\right)^T$ for some $\rho > 0$ and for all $j > N$.
Hence

$$\left[\frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi_i(a)|)\|}{\rho}\right)\right]^{r_i}\right]^{1/T} \leq \frac{\epsilon}{2}$$

for some $\rho > 0$ and for all $j > N$.

Now let $0 < |\lambda| < 1$. Since each $M_i$ satisfies $\Delta_2$-condition (i.e.,$M_i(Lx) \leq L\, M_i(x), x \geq 0, L < 1$ ) and by **Inequality 3**., we have for all $j > N$ we get

$$\frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\lambda\phi_i(a)|)\|}{\delta}\right)\right]^{r_i}$$
$$\leq \left[1 + [|\lambda|]\right]^H \times$$
$$\frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\phi_i(a)|)\|}{\delta}\right)\right]^{r_i} < \left(\frac{\epsilon}{2}\right)^T$$

Since $M_i$ is continuous everywhere in $[0, \infty[$
then for $j \leq N$,
$\psi(t) = \frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|t\phi_i(a)|)\|}{\delta}\right)\right]^{r_i}$
is continuous at 0. So, there is a $\delta (0 < \delta < 1)$
such that $|\psi(t)| < \left(\frac{\epsilon}{2}\right)^T$ for $0 < t < \delta$.
Let there exist B such that $|\lambda| < \delta$ for $n > B$.
Hence for $n > B$ and $j \leq N$,

$$\left[\frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\lambda\phi_i(a)|)\|}{\rho}\right)\right]^{r_i}\right]^{1/T} < \frac{\epsilon}{2}$$

for $j = 1, 2, \ldots\ldots, N$.
Thus

$$\left[\frac{1}{h_j}\sum_{i\in I_j} i^{-s}\left[M_i\left(\frac{\|f_i(|\lambda\phi_i(a)|)\|}{\rho}\right)\right]^{r_i}\right]^{1/T} < \epsilon$$

for all $n > B$ and for all $j$ and hence $g(\lambda a) \to 0$ *as* $\lambda \to 0$.
Consequetly g becomes a paranorm function on the space $|\overline{N_p}|_\theta^0(X_{\phi(a)}, M, F, r, s)$ becomes a paronormed pace.
We now study Lacunary statistical convergence method with respect to $|\overline{\mathbf{N}}, \mathbf{p_k}|$- summability [5] as follows:

**Lacunary Statistical Convergence with respect $|\overline{\mathbf{N}}, \mathbf{p_k}|$- Summability on $|\overline{\mathbf{N_p}}|_\theta^0(X_{\phi(a)}, \mathbf{M}, \mathbf{F}, \mathbf{r}, s)$:**

A sequence $a = (a_n)$ is $|\bar{S}|_\theta$ − statistically convergent to $\ell \in X$ with respect to $|\bar{\mathbf{N}}, \mathbf{p_k}|$- summability, if for any $\epsilon > 0$,

$$\lim_{j\to\infty} \frac{1}{h_j}\left|\{i \in I_j : \|f_i(|\phi_i(a) - \ell|)\| \geq \varepsilon\}\right| = 0$$

The space of all $|\bar{S}|_\theta$-lacunary statistically convergent with respect to $|\bar{\mathbf{N}}, \mathbf{p_k}|$summability is denoted by $|\bar{S}|_{\theta,|\bar{\mathbf{N}},\mathbf{p_k}|}$.

**Theorem 4.1** If $\phi_i(a) \to \ell$, then $a = (a_n) \in |\bar{S}|_{\theta,|\bar{\mathbf{N}},\mathbf{p_k}|}$ .

**Proof:** Let $\phi_i(a) \to \ell$. Then by definition of modulus function it follows that $f_i(|\phi_i(a) - \ell|) \to 0$ as $i \to \infty$ and consequently, $\|f_i(|\phi_i(a) - \ell|)\| \to 0$ since $X$ is a $K$-space. Hence $a = (a_n) \in |\bar{S}|_{\theta,|\bar{\mathbf{N}},\mathbf{p_k}|}$.

**Theorem 4.1** $\left|\overline{\mathrm{N}_\mathrm{p}}\right|_\theta\left(X_{\phi(a)}, \mathrm{M}, \mathrm{F}, \mathrm{r},\right) \subset |\bar{\mathrm{S}}|_{\theta,\ |\bar{N}, p_k|}$.

**Proof:** Let $x = (x_i) \in \left|\overline{\mathrm{N}_\mathrm{p}}\right|_\theta\left(X_{\phi(a)}, \mathrm{M}, \mathrm{F}, \mathrm{r}\right)$

Then $\exists\ \ell \in \mathbb{C}$ such that

$$\lim_{j\to\infty} \frac{1}{h_j}\sum_{i\in I_j}\left[M_i\left(\frac{\|f_i(|\phi_i(a)| - \ell)\|}{\rho}\right)\right]^{r_i} = 0$$

for some $\rho > 0$.

Now, $\frac{1}{h_j}\sum_{i\in I_j}\left[M_i\left(\frac{\|f_i(|\phi_i(a)|-\ell)\|}{\rho}\right)\right]^{r_i} = \frac{1}{h_j}\sum_{\substack{i\in I_j\\ \|f_i(|\phi_i(a)|-\ell)\|\geq\varepsilon}}\left[M_i\left(\frac{\|f_i(|\phi_i(a)|-\ell)\|}{\rho}\right)\right]^{r_i}$

$$+\frac{1}{h_j}\sum_{\substack{i\in I_j\\ \|f_i(|\phi_i(a)|-\ell)\|<\varepsilon}}\left[M_i\left(\frac{\|f_i(|\phi_i(a)| - \ell)\|}{\rho}\right)\right]^{r_i}$$

$$\geq \frac{1}{h_j}\sum_{\substack{i\in I_j\\ \|f_i(|\phi_i(a)|-\ell)\|\geq\varepsilon}}\left[M_i\left(\frac{\|f_i(|\phi_i(a)| - \ell)\|}{\rho}\right)\right]^{r_i}$$

$$\geq \frac{1}{h_j}\sum_{\substack{i\in I_j\\ \|f_i(|\phi_i(a)|-\ell)\|\geq\varepsilon}}[M_i(\epsilon)]^{r_i}$$

$$\geq \frac{1}{h_j}\sum_{\substack{i\in I_j\\ \|f_i(|\phi_i(a)|-\ell)\|\geq\varepsilon}}\min\left(\begin{matrix}[M_i(\varepsilon)]^{\inf r_i},\\ [M_i(\varepsilon)]^{\sup r_i}\end{matrix}\right)$$

$$\geq \frac{1}{h_j}\left|\{i \in I_j : \|f_i(|\phi_i(a)| - \ell)\| \geq \varepsilon\}\right| \quad\times\quad \min\left([M_i(\varepsilon)]^{\inf r_i},\ [M_i(\varepsilon)]^{\sup r_i}\right)$$

Taking limit $r \to \infty$ on both sides the result follows.

**Remark 5.1** The inclusion is strict.

**Example:** Suppose that $M_i = M$ is unbounded and $r_i = 1$ for each $i$ and $\theta = (k_r)$ be a lacunary sequence, so that there exist a positive sequence $(y_{k_m})$ such that $M\left(\frac{y_{k_m}}{\rho}\right) = h_m^2$ for some $\rho > 0$ and $m = 1, 2, \ldots$. Also we define, $\|f_i(|\phi_i(a)|)\| = \begin{cases} y_{k_m}, & i = m^2 \\ \theta, & \text{otherwise} \end{cases}$.

Then we have, $\frac{1}{h_j}\left|\{i \in I_j : \|f_i(|\phi_i(a)|)\| \geq \varepsilon\}\right| \leq \frac{[\sqrt{h_j}]}{h_j} \to 0$ as $j \to \infty$

Consequently, $a = (a_n) \xrightarrow{|\bar{\mathrm{S}}|_{\theta,\ |\bar{N}, p_k|}} 0$.

But, $\lim_{j\to\infty}\frac{1}{h_j}\sum_{i\in I_j}\left[M_i\left(\frac{\|f_i(|\phi_i(a)|)\|}{\rho}\right)\right]^{r_i} \neq 0$.

Hence $a = (a_n) \notin \left|\overline{\mathrm{N}_\mathrm{p}}\right|_\theta\left(X_{\phi(a)}, \mathrm{M}, \mathrm{F}, \mathrm{r},\right)$.

**Theorem 5.1:** $\left|\overline{\mathrm{N}_\mathrm{p}}\right|_\theta\left(X_{\phi(a)}, \mathrm{M}, \mathrm{F}, \mathrm{r},\right) = |\bar{\mathrm{S}}|_{\theta,\ |\bar{N}, p_k|}$iff $M_i$ is bounded for each $i$.

**Proof:** Suppose that $M_i$ is bounded for each $i$ and $a = (a_n) \xrightarrow{|\bar{\mathrm{S}}|_{\theta,\ |\bar{N}, p_k|}} 0$. Hence there exist an integer $K$ such that $M_i(t) < K$, for each $i$ and each $t \geq 0$. Then for each $j$,

$$\frac{1}{h_j}\sum_{i\in I_j}\left[M_i\left(\frac{\|f_i(|\phi_i(a)|)\|}{\rho}\right)\right]^{r_i} = \frac{1}{h_j}\sum_{\substack{i\in I_j \\ \|f_i(|\phi_i(a)|)\|\geq\varepsilon}}\left[M_i\left(\frac{\|f_i(|\phi_i(a)|)\|}{\rho}\right)\right]^{r_i} + \frac{1}{h_j}\sum_{\substack{i\in I_j \\ \|f_i(|\phi_i(a)|)\|\geq\varepsilon}}\left[M_i\left(\frac{\|f_i(|\phi_i(a)|)\|}{\rho}\right)\right]^{r_i}$$

$$\geq K\,\frac{1}{h_j}\left|\{i\in I_j : \|f_i(|\phi_i(a)|)\|\geq\varepsilon\}\right| + \min\left([M_i(\varepsilon)]^{\inf r_i},\ [M_i(\varepsilon)]^{\sup r_i}\right)$$

Now, taking limit as $j\to\infty$ the result follows. The converse follows from **Theorem 4.2.** Hence the proof.

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